CHARACTERIZATION AND CLASSIFICATION OF MINUSCULE KAC–MOODY REPRESENTATIONS BUILT FROM COLORED POSETS

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ABSTRACT

Michael C. Strayer: Characterization and classification of minuscule Kac–Moody representations built from colored posets
(Under the direction of Robert A. Proctor)

Colored minuscule and *d*-complete partially ordered sets encode information that can be used to construct many objects that arise in semisimple and Kac–Moody Lie theory. The colored minuscule posets can be used to construct the minuscule representations of the semisimple Lie algebras. R.M. Green extended this picture with his full heap colored posets, showing they are sufficient to construct beautiful "doubly infinite" representations of affine Kac–Moody algebras. We reformulate and build upon his work. We obtain necessary poset coloring conditions for the construction of representations of Kac–Moody algebras that uniformly incorporate the minuscule representations of semisimple Lie algebras and Green's representations of the affine algebras. While doing so, we delineate which defining relations for the algebras correspond to which coloring properties for the posets and obtain representations of the Borel subalgebras as well. This leads to the development of new "frontier census" coloring properties and new uniform definitions of four kinds of colored posets: Finite and infinite minuscule and *d*-complete posets. This is the first definition of an infinite colored *d*-complete poset. Building upon work of R.A. Proctor, J.R. Stembridge, R.M. Green, and Z.S. McGregor-Dorsey, we classify these posets and their associated Dynkin diagrams. This in turn classifies all minuscule representations that can be built from colored posets.

To Rachelle and Hannah.
I am excited to share the future with you both.

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CHAPTER 1

Introduction and background

1.1 Introduction

The minuscule representations of the semisimple Lie algebras are the irreducible highest weight representations whose weights are all in the Weyl group orbit of the highest weight. Under the standard ordering on the set of weights, the weight diagrams of these representations are distributive lattices. The posets of join irreducible elements of these distributive lattices of weights are known as the "minuscule" posets. Their elements can be "colored" with the nodes of the associated Dynkin diagram. These colorings can be used to represent the actions of the Chevalley generators of the algebras on the spaces generated by the "order ideals" of these colored minuscule posets.

This dissertation is concerned with extending this picture in two ways. First, we extend from semisimple Lie algebras to arbitrary Kac–Moody algebras. Second, we consider representations of their Borel subalgebras as well as representations of the Kac–Moody algebras. After presenting definitions in Chapter 2, this dissertation has two parts. Part I consists of Chapters 3–6, wherein we produce an axiomatic characterization of when certain analogous "minuscule" representations of the above algebras can be built from colored posets. Part II consists of Chapters 7–8, wherein we classify the colored posets satisfying the axiomatic characterization from Part I. This in turn classifies the associated representations built from colored posets.

To extend the notion of building minuscule representations from colored posets to the more general Kac–Moody settings, we respectively define "P-minuscule" and "upper P-minuscule" representations in Section 2.3. Those definitions are each made with respect to a given poset P that is colored by the nodes of the Dynkin diagram Γ that specifies the Kac–Moody algebra at hand. *A priori* it is not assumed whether the dimensions of these representations are finite or infinite. Distinguishing between these two cases, we can picture our four settings in the 2×2 table below, Table 1.1.

The central problem this dissertation solves is to describe the colored posets that fill in this table: Which colored posets produce the finite and infinite dimensional P-minuscule and upper P-minuscule representations? Parts I and II each provide a filling of Table 1.1. In Part I we develop uniform cardinality-

Representation	**Finite dimensional**	**Infinite dimensional**
***P*-minuscule**		
Upper *P*-minuscule		

Table 1.1: Our four settings for minuscule representations built by colored posets

independent definitions of two types of colored posets that axiomatically characterize the P-minuscule and upper P-minuscule representations. The main results of Part I state these axiomatic characterizations. See Theorems 5.1.1 and 5.2.2 for the simply laced case and Theorems 5.3.1 and 5.4.2 for the general case. We define the "Γ-colored minuscule" and "Γ-colored d-complete" posets to be the colored posets that respectively satisfy these two collections of axioms. Using this shorthand terminology, these characterizations are summarized in Theorem 6.1.1. Specializing these posets to the finite and infinite cases fills in the table:

Representation	**Finite dimensional**	**Infinite dimensional**
***P*-minuscule**	Finite Γ-colored minuscule posets	Infinite Γ-colored minuscule posets
Upper *P*-minuscule	Finite Γ-colored d-complete posets	Infinite Γ-colored d-complete posets

Table 1.2: The characterizations of P-minuscule and upper P-minuscule representations

In Part II, working only with these collections of axioms, we classify the Γ-colored minuscule and Γ-colored d-complete posets. To do this, we refer to earlier classification results obtained by R.A. Proctor, J.R. Stembridge, R.M. Green, and Z.S. McGregor-Dorsey. Here the main results are stated in Theorems 8.3.8 and 8.4.5. Table 1.3 previews how the 2×2 table will be filled in with lists of connected posets.

Connected posets	**Finite**	**Infinite**
Γ-colored minuscule	Colored minuscule posets	Full heaps
First introduced by:	R.A. Proctor (1984)	R.M. Green (2007)
Γ-colored d-complete	Dominant minuscule heaps	Filters of full heaps
First introduced by:	J.R. Stembridge (2001)	This dissertation (2019)

Table 1.3: The classifications of connected Γ-colored minuscule and Γ-colored d-complete posets

Finally, we present the complete description of the P-minuscule and upper P-minuscule representations in Theorem 8.5.1. It is an immediate consequence of our main results listed above.

Section 1.2 is a background section. There we briefly describe the work of Proctor, Stembridge, Green, and McGregor-Dorsey; more detail for this is added in Chapter 7. We will describe our new developments and contributions to this area in Section 1.3. Section 1.4 expands upon the table of contents, describing the progression of results in this dissertation.

1.2 Pre-existing minuscule and d-complete posets, dominant minuscule and full heaps

Here we describe how this subject developed from 1980 to 2013. The principal antecedents to this dissertation are [Gr1], [Gr3], and [McG], and important antecedents to those works were [Pr1], [Pr4], [Ste] and [Ha2]. Those references were concerned with colored minuscule posets, colored d-complete posets and the closely related dominant minuscule heaps, and full heaps. In this section we indicate where these kinds of posets first appeared. These are the posets that appear in Table 1.3.

It has been known that the minuscule representations of the semisimple Lie algebras can be constructed combinatorially. Consider one of the irreducible minuscule posets P introduced by Proctor that were colored by him in Theorem 11 of [Pr1] with the nodes of an associated Dynkin diagram Γ. These are the "colored minuscule posets" of Table 1.3. Form the set $\mathcal{F}\mathcal{I}(P)$ of all of the "splits" F/I of P: Here F is an upwardly closed subset (filter) of P and I is its complementary downwardly closed subset (ideal) of P. These splits are the elements of a Bruhat distributive lattice, called an irreducible minuscule lattice in [Pr1], whose covering edges are colored by the Dynkin nodes. These edges can be used to define colored raising and lowering actions of the Chevalley generators of the Lie algebra $\mathfrak{g}$ associated to Γ. It can be seen that these actions specify a representation of $\mathfrak{g}$ "carried by" $\mathcal{F}\mathcal{I}(P)$. Wildberger used [Wil] this picture to specify the actions in a minuscule representation of a Chevalley basis for the Lie algebra; see Section 7.2 of [Gr3]. R.G. Donnelly constructed many representations of semisimple Lie algebras using lattices of splits; see [Don] and subsequent papers. Proctor showed [Pr1] that minuscule posets have the combinatorial Sperner property and (with R. Stanley's help) the combinatorial Gaussian property; see Section 11.3 of [Gr3]. Minuscule posets are the structures on which the Littlewood–Richardson and cohomology calculations for minuscule varieties performed in [BuSa] and its references are based. The minuscule posets are classified by the minuscule highest weights of the associated representations.

A generalization of minuscule posets appeared after Dale Peterson introduced [Car] a special kind of element in the Kac–Moody Weyl group specified by a Dynkin diagram Γ. For an integral weight λ, he defined "λ-minuscule" elements w. When Γ is simply laced, Proctor showed [Pr4] that the Bruhat intervals

$[e, w]$ are distributive lattices. When λ is dominant, he then characterized the finite Γ-colored poset P of join irreducibles of the lattice with some structural "d-complete" conditions. The reduced decompositions of w corresponded to the linear extensions of P. Working in the context of Viennot's heap for w, Stembridge extended [Ste] Proctor's work to include non-simply laced Γ. He reformulated Proctor's notion of colored d-complete with some elegant coloring axioms, and referred to these posets as "dominant minuscule heaps." More generally, Stembridge characterized the heaps for all λ-minuscule elements. M. Hagiwara described [Ha1, Ha2] the minuscule heaps for elements of the Kac–Moody Weyl groups specified by star shaped Dynkin diagrams and for the affine Weyl group of type $\tilde{A}_n$. Stembridge's coloring axioms for the d-complete posets are not all self-dual. Proctor showed that d-complete posets have unique jeu de taquin rectifications [Pr5] and (with D. Peterson's help) the hook length property [Pr6]. These posets have been receiving increasing attention, as in [KlRa]. There is a bibliography for them in [PrSc]. When that study of the axioms for finite uncolored d-complete posets was written, it became apparent that the definition of "d-complete" could likely be extended to infinite locally finite posets. But it was unclear precisely what the most appropriate definition should be for such posets. For further historical details, see Section 13 of [Pr6].

Proctor classified [Pr3] the uncolored d-complete posets as "slant sums" of "slant irreducible" uncolored d-complete posets. He organized the possible slant irreducible d-complete posets into fifteen explicitly drawn families. An uncolored d-complete poset may be colored essentially uniquely [Pr4, Prop. 8.6] to form a colored d-complete poset, so this classification can be applied to colored d-complete posets as well. When Stembridge extended Proctor's work to include the non-simply laced case, he organized [Ste] the additional possibilities into two additional families.

Adopting some of Stembridge's axioms, Green axiomatically defined [Gr1] "full heaps" colored by Dynkin diagrams Γ. These are infinite locally finite colored posets P in which the appearances of each color from Γ are unbounded above and below; we refer to such structures as "doubly infinite." He regarded these posets as being close companions to the finite minuscule posets of [Pr1]. All of his coloring axioms for full heaps were self-dual. The "extended slant lattices" used by Hagiwara to describe the minuscule heaps for type $\tilde{A}_n$ were early appearances of full heaps.

R. M. Green used full heaps to construct [Gr1] a small number of beautiful doubly infinite representations of most affine Kac–Moody algebras; their weight diagrams were unbounded above and below. In contrast, the familiar Category O representations have weight diagrams that are bounded above. Green's representations and the full heaps from which they were built formed a central topic in his 2013 Cambridge tract [Gr3]. He noted that these representations (with no highest weights) are analogous in many ways to the minuscule representations of semisimple Lie algebras (which are finite dimensional with highest weights). For infinite dimensional Kac–Moody algebras, there are no highest weight representations that are analogous to the minuscule representations of semisimple Lie algebras. This dissertation launches a program in which an abstract notion of "minuscule" representation for arbitrary Kac–Moody algebras will be defined. These representations will also be classified. See Section 9.3.

After introducing full heaps, Green defined [Gr3] the notion of "principal subheap" for full heaps colored by affine Γ. These are finite colored posets. He showed that the principal subheaps of such a full heap are isomorphic to each other. Then he proved that the possible principal subheaps were exactly the pre-existing finite colored minuscule posets. Our classification in Section 8.4 shows that Green's full heaps are exactly our infinite Γ-colored minuscule posets and that Green's principal subheaps (the pre-existing finite colored minuscule posets) are exactly our finite Γ-colored minuscule posets. The relationship between the finite Γ-colored minuscule posets and the infinite Γ-colored minuscule posets is entirely different here than in [Gr3].

Green classified [Gr3, Thm. 6.6.2] the full heaps colored by Dynkin diagrams of affine type. His student Z.S. McGregor-Dorsey then showed [McG, Thm. 4.7.1] that any Dynkin diagram with finitely many nodes that colors a full heap must consist of affine connected components. Together these results showed that the list in Theorem 6.6.2 of Green was already the complete list of full heaps colored by connected Dynkin diagrams.

1.3 Our development of Γ-colored d-complete and Γ-colored minuscule posets

The unified definitions that we develop for finite and infinite "Γ-colored minuscule" and "Γ-colored d-complete" posets may be of more interest than any one of our stated results by itself. These new overarching definitions do not depend on knowing the cardinality of the poset *a priori*; in addition they provide the first definition of an infinite colored d-complete poset. These definitions are presented in Section 6.1, wherein our main results of Part I are also summarized. That section has been written for immediate accessibility.

It is the introduction of "frontier census" coloring properties for a poset P that enables us to provide definitions of "Γ-colored minuscule" and "Γ-colored d-complete" that uniformly fill Table 1.2. Given an element $y \in P$ that is an extreme element of P of color $b \in \Gamma$, these properties limit the number of elements that lie beyond y in P that have colors that are adjacent to b in Γ. The frontier census properties are presented in Section 4.3 in the simply laced case and are extended to the general case in Section 4.6.

Let $\mathfrak{g}'$ be the derived Kac–Moody algebra for a Dynkin diagram Γ. Green showed in Theorem 4.1.6(i) of

[Gr3] that the lattice of splits of a full heap for Γ "carried" a representation of $\mathfrak{g}'$; this first appeared as Theorem 3.1 of [Gr1]. The "necessary" direction of our Theorem 5.4.2 provides a converse to Theorem 4.1.6(i) that includes finite dimensional representations as well as infinite dimensional representations. To state this converse, we formulate a notion of "P-minuscule" representation (Section 2.3): This is a representation of $\mathfrak{g}'$ carried by $\mathcal{FI}(P)$ that "looks like" a minuscule representation of a semisimple Lie algebra. At the same time we obtain a version of the "sufficient" Theorem 4.1.6(i) of [Gr3] that now includes posets of unknown (finite or "mixed") cardinality. While doing this we produce several intermediate results: As more and more coloring properties are assumed for the poset P, the representations constructed have stronger and stronger algebraic properties. Most often these algebraic properties are the satisfaction of some of the defining relations for $\mathfrak{g}'$. Each of these collections of coloring properties is necessary as well as being sufficient for the collection of algebraic properties at hand. This development clarifies which collections of the coloring properties assumed in Theorem 4.1.6(i) correspond to which algebraic aspects of the representations. It also facilitates comparison with the collections of coloring properties considered by Stembridge in his parallel study of the reduced decompositions of a λ-minuscule Kac–Moody Weyl group element w.

By omitting the "down only" coloring property required in Theorem 5.4.2, earlier in Theorem 5.3.1 we obtain a similar characterization of colored poset constructions of representations of just the "Borel derived" (Section 2.2) subalgebra $\mathfrak{b}'_+$. When P is finite, these representations arise [Str, §14] as the restrictions to $\mathfrak{b}'_+$ of the Demazure $\mathfrak{b}_+$-modules for the dominant λ-minuscule w. (Even when P is finite, the Kac–Moody algebra $\mathfrak{g}$ at hand can be infinite dimensional.) For this "up only" analog to Theorem 5.4.2, we introduce two new definitions. We formulate the notion of "upper P-minuscule" representation of $\mathfrak{b}'_+$ (Section 2.3). For the lower right corner of Table 1.1, we formulate a notion of colored d-complete that works for infinite locally finite colored posets. So this desire to obtain a theorem for $\mathfrak{b}'_+$ analogous to Theorem 5.4.2 led to the first definition for infinite colored d-complete posets.

Proctor classified [Pr3, Pr4] the finite colored d-complete posets for simply laced Γ. Stembridge extended [Ste] this classification to non-simply laced Γ. Green [Gr3] and McGregor-Dorsey [McG] classified the full heaps. In Part II of this dissertation we classify the infinite colored d-complete posets for simply laced Γ and list all of the posets that are organized by Table 1.2. Having the new "necessary" direction of Theorem 5.4.2 available will enable us to also classify the P-minuscule representations. This will be done by using that direction to combine two of the main results of [Gr3] and [McG], namely Theorem 4.1.6(i) and their classification of full heaps. This classification of P-minuscule representations will be a step in our minuscule

Kac–Moody program: In an anticipated paper a definition of "abstract minuscule" representation for a Kac–Moody algebra will be presented. This definition will not refer to a poset that has been supplied *a priori*. Given such a representation, it should be possible to construct a poset P so that the given representation can be viewed as a P-minuscule representation. We preview this proposed new definition in Section 9.3.

1.4 Overview and organization

After definitions are given in Chapter 2, this dissertation has two parts. Part I consists of Chapters 3–6, and Part II consists of Chapters 7–8. We obtain the axiomatic characterization of upper P-minuscule and P-minuscule representations in Part I, and we classify the posets that satisfy this axiomatic description in Part II. Chapter 9 contains some further remarks.

Each of Chapters 3, 4, and 5 begin with two or three sections which develop the simply laced case; then later parallel sections extend those developments to the general case. This extending process is described in more detail in Section 2.5.

Chapter 3 concerns representations of the Borel derived subalgebra $\mathfrak{b}'_+$ that are carried by the lattice of splits $\mathcal{FI}(P)$. Theorem 3.1.4 states that the possession of three of our earliest coloring properties by the poset P is equivalent to the existence of a representation of the smaller subalgebra $\mathfrak{n}_+ \subset \mathfrak{b}'_+$ that is carried by $\mathcal{FI}(P)$. Section 3.2 studies the extension of this representation of $\mathfrak{n}_+$ to the Borel derived subalgebra $\mathfrak{b}'_+$ by specifying the actions of the simple coroots (which form a basis of the Cartan derived subalgebra $\mathfrak{h}'$). There is some freedom available for such an extension; the weight functions we introduce are accounting tools to keep track of the coroot actions.

In Chapter 4, we introduce a particular nice weight function. The prototypical minuscule representations of semisimple Lie algebras have weights along their "$\mathfrak{sl}_2$ strings" that are composed of eigenvalues from $\{-1, 0, +1\}$ for the simple coroot actions. Our preferred weight function is defined in Section 4.1. In Proposition 4.2.1 we begin to obtain simple coroot actions with $\{-1, 0, +1\}$ values when three coloring properties for P are present. We introduce the frontier census coloring properties in Section 4.3.

We obtain our main results of Part I in Chapter 5 when we axiomatically characterize the upper P-minuscule and P-minuscule representations. We summarize these results in Chapter 6 by presenting our new definitions of Γ-colored d-complete and Γ-colored minuscule posets.

We then shift our focus to the classifications of Γ-colored d-complete and Γ-colored minuscule posets. This classification takes place in the general case. In Chapter 7, we state the prerequisite definitions and results needed for the classification. We also give equivalences between various sets of our coloring properties with sets of Stembridge and Green in Section 7.4. We obtain the classifications of Γ-colored d-complete and Γ-colored minuscule posets in Chapter 8. Theorem 8.5.1 is the culmination of this dissertation; it answers the central question posed in Section 1.1 by filling in Table 1.1 with explicit lists of colored posets.

CHAPTER 2

Definitions

In this chapter we give the most routine definitions we will need; more specialized definitions that have been developed for this dissertation will be introduced when needed. Definitions in the first three sections are made within the context of the simply laced case. In Section 2.4 we extend definitions to the general case. Then in Section 2.5 we explain how the extensions of the simply laced results to the general case are organized in Chapters 3–5.

2.1 Combinatorial definitions

Fix a partially ordered set P throughout. Letters such as $z, y, x, \ldots$ are used to denote elements of P. Consult [Sta] for the following terminology: comparable elements, covering relations and the Hasse diagram, chains and saturated chains, antichains, closed and open intervals, connected posets, direct sums of posets, join irreducible elements, convex subsets, linear extensions, order dual poset P^*, and rank. We assume each P is *locally finite*; this means that all of its closed intervals are finite. We write $x \to y$ to indicate that x is covered by y. We say x and y are *neighbors* in P if $x \to y$ or $y \to x$. A subset $F \subseteq P$ is a *filter* of P if whenever $x \in F$ and $y \geq x$, we also have $y \in F$. Dually, a subset $I \subseteq P$ is an *ideal* of P if whenever $x \in I$ and $y \leq x$, we also have $y \in I$. For each filter F of P there is a corresponding ideal $I := P - F$. Let $\mathcal{FI}(P)$ be the set of all ordered pairs (F, I) such that F is a filter of P and I is its corresponding ideal: These are the *splits* of P. The set $\mathcal{FI}(P)$ becomes a distributive lattice when it is ordered by inclusion of the ideals within the splits. Figures 2.1 and 2.2 display posets P and their lattices $\mathcal{FI}(P)$ of splits. We write $(F + x + \ldots, I - x - \ldots)$ instead of $(F \cup \{x, \ldots\}, I - \{x, \ldots\})$ and $(F - x - \ldots, I + x + \ldots)$ instead of $(F - \{x, \ldots\}, I \cup \{x, \ldots\})$. Each edge in the Hasse diagram of $\mathcal{FI}(P)$ can be viewed as transferring a minimal element of some split's filter to its ideal, where it becomes a maximal element: In $\mathcal{FI}(P)$ one has $(F, I) \to (F - x, I + x)$ when x is a minimal element of F. Dually, one has $(F + y, I - y) \to (F, I)$ when y is a maximal element of I.

Fix a finite simple graph Γ: No loops or multiple edges are allowed. We will use the symbol Γ to also denote its set of vertices. Letters such as $a, b, c, \ldots$ are used to denote vertices of Γ, which we call *colors*. A

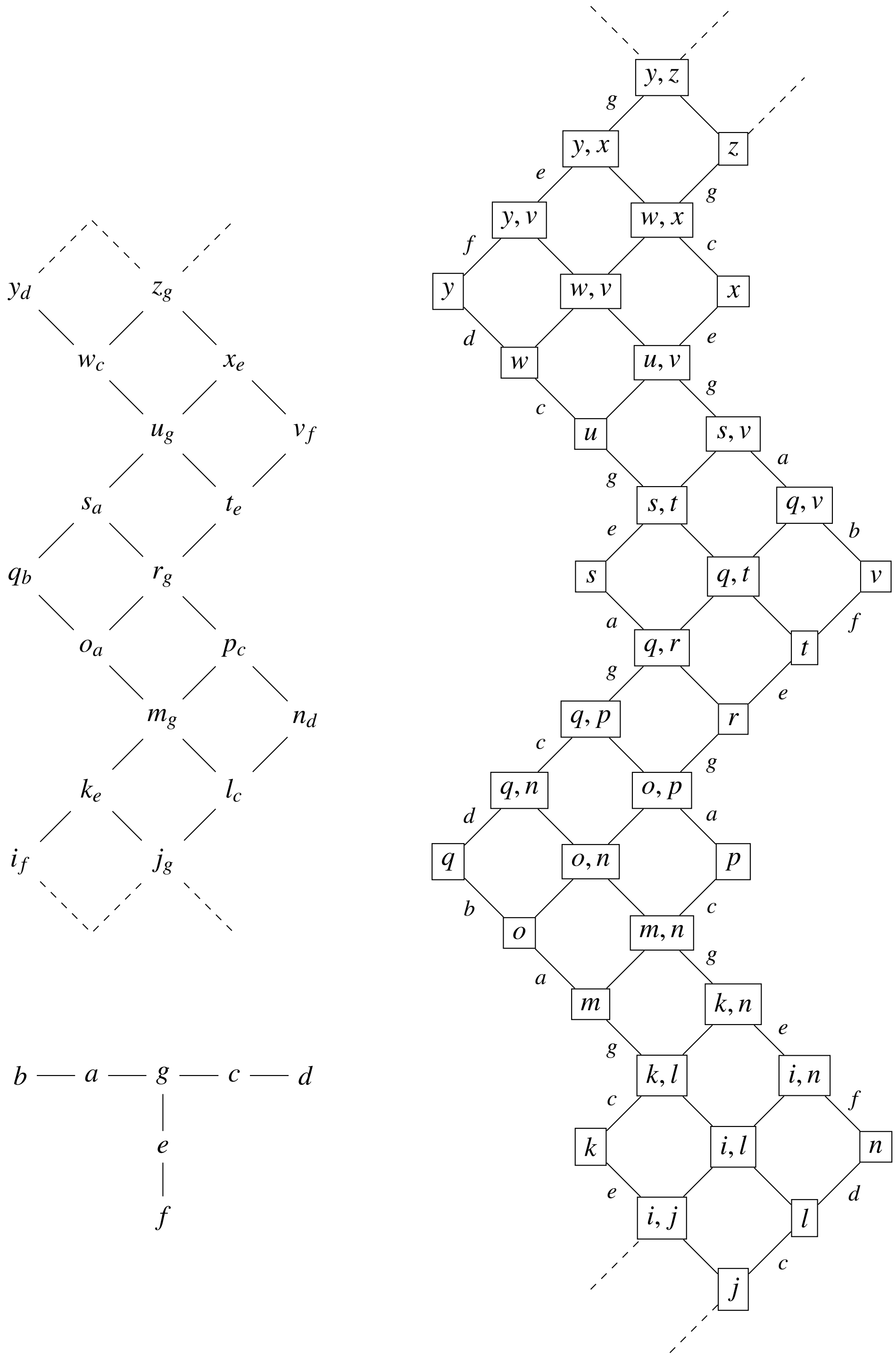


Figure 2.1: Clockwise from bottom left: Simple graph Γ of affine type $\tilde{E}_6$, full heap poset P colored by Γ, and edge-colored lattice of filter-ideal splits $\mathcal{FI}(P)$. Subscripts in P indicate element colors. Splits (F, I) are described by the maximal elements of the ideal I. Opposite diamond edges in $\mathcal{FI}(P)$ have the same color.

Γ-*set* is a set of entities indexed by the colors in Γ. Let $a, b \in \Gamma$. If $\{a, b\}$ is an edge of Γ, we write $a \sim b$ and say a and b are *adjacent*. If $a \neq b$ and $\{a, b\}$ is not an edge of Γ, we write $a \nsim b$ and say a and b are *distant*. Let δ_{ab} be the Kronecker delta: Here $\delta_{ab} := 1$ if $a = b$ and $\delta_{ab} := 0$ otherwise. Let γ_{ab} be the adjacency indicator: Here $\gamma_{ab} := 1$ if $a \sim b$ and $\gamma_{ab} := 0$ otherwise. Note that $\gamma_{ab} = \sum_{c \sim b} \delta_{ac}$. Define $\theta_{ab} := 2\delta_{ab} - \gamma_{ab}$. We have $\theta_{ab} = 2$ if $a = b$ and $\theta_{ab} = -1$ if $a \sim b$ and $\theta_{ab} = 0$ if $a \nsim b$.

We equip P with a surjective *coloring function* $\kappa : P \to \Gamma$. We say that P is a Γ-*colored* poset. See Figures 2.1 and 2.2. For each $a \in \Gamma$, let $P_a := \kappa^{-1}(a)$ be the subset of all elements in P of color a. The coloring of P induces an edge coloring of the Hasse diagram of $\mathcal{FI}(P)$: The color of an edge is given by the color of the element transferred along that edge.

Various poset coloring properties will be precisely defined as needed; Table 2.1 indexes these forthcoming definitions. The poset displayed in Figure 2.1 satisfies all of these properties; the poset displayed in Figure 2.2 satisfies all of them except Mn1LA. See also Table 2.4 in Section 2.4, which indexes the extensions of the properties I3ND, I2A, MxkGA, and MnkLA to the general case.

Property:	**Abbreviated definition:**	**Location:**
EC	elements with Equal colors are Comparable	Proposition 3.1.1
ND	Neighbors have Different colors	Proposition 3.1.1
NA	Neighbors have Adjacent colors	Lemma 3.1.2
I3ND	Interval of 3 Neighbors has 3 Different colors	Lemma 3.1.3
AC	elements with Adjacent colors are Comparable	Proposition 4.1.2
I2A	consecutive color Intervals contain 2 Adjacent colors	Proposition 4.1.2
MxkGA	color Max has $\leq k$ elts Greater than it w/ Adjacent colors	Section 4.3
MnkLA	color Min has $\leq k$ elts Less than it w/ Adjacent colors	Section 4.3

Table 2.1: Coloring property abbreviations and locations of definitions; also see Table 2.4.

2.2 Algebraic definitions

We regard the graph Γ as being a *simply laced Dynkin diagram*: Once Γ has been given a total ordering, the associated generalized Cartan matrix is $[\theta_{ab}]_{a,b\in\Gamma}$. The *Kac–Moody algebra* $\mathfrak{g}$ with *Cartan subalgebra* $\mathfrak{h}$ and subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are defined in [Kac] for a given Dynkin diagram Γ. The *Weyl group* of $\mathfrak{g}$ is also defined in [Kac]. The *positive* and *negative Borel subalgebras* are respectively $\mathfrak{b}_\pm := \mathfrak{h} + \mathfrak{n}_\pm$. We refer to the subalgebras of the *derived subalgebra* $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ formed by intersections of $\mathfrak{h}$ and $\mathfrak{b}_\pm$ with $\mathfrak{g}'$ as the *Cartan derived subalgebra* $\mathfrak{h}'$ and the *Borel derived subalgebras* $\mathfrak{b}'_\pm$. The algebras $\mathfrak{n}_\pm$ and the derived algebras are generated [Kac, §9.11] by subsets of the symbols $\{x_a, y_a, h_a\}_{a\in\Gamma}$ subject to some *defining relations*. When Γ is simply laced, the relations are certain unions (specified below) of the following sets:

$$
\begin{array}{lrl}
\text{(XX):} & \text{(i)} & [x_b, x_a] = 0 \text{ if } a, b \in \Gamma \text{ with } a \not\sim b,\\
 & \text{(ii)} & [x_a, [x_a, x_b]] = 0 \text{ for all } a, b \in \Gamma,\\[4pt]
\text{(YY):} & \text{(i)} & [y_b, y_a] = 0 \text{ if } a, b \in \Gamma \text{ with } a \not\sim b,\\
 & \text{(ii)} & [y_a, [y_a, y_b]] = 0 \text{ for all } a, b \in \Gamma,\\[4pt]
\text{(HH):} & \text{(i)} & [h_b, h_a] = 0 \text{ for all } a, b \in \Gamma,\\[4pt]
\text{(HX):} & \text{(i)} & [h_a, x_a] = 2x_a \text{ for all } a \in \Gamma,\\
 & \text{(ii)} & [h_b, x_a] = -x_a \text{ if } a, b \in \Gamma \text{ with } a \sim b,\\
 & \text{(iii)} & [h_b, x_a] = 0 \text{ if } a, b \in \Gamma \text{ with } a \not\sim b,\\[4pt]
\text{(HY):} & \text{(i)} & [h_a, y_a] = -2y_a \text{ for all } a \in \Gamma,\\
 & \text{(ii)} & [h_b, y_a] = y_a \text{ if } a, b \in \Gamma \text{ with } a \sim b,\\
 & \text{(iii)} & [h_b, y_a] = 0 \text{ if } a, b \in \Gamma \text{ with } a \not\sim b,\\[4pt]
\text{(XY):} & \text{(i)} & [x_a, y_a] = h_a \text{ for all } a \in \Gamma,\\
 & \text{(ii)} & [x_b, y_a] = 0 \text{ if } a, b \in \Gamma \text{ with } a \neq b.
\end{array}
$$

The relations HX and HY can be condensed to $[h_b, x_a] = \theta_{ab}x_a$ and $[h_b, y_a] = -\theta_{ab}y_a$ for $a, b \in \Gamma$. The algebra $\mathfrak{h}'$ is the Lie algebra generated by $\{h_a\}_{a\in\Gamma}$ subject to the relation HH, and so it is abelian. The algebra $\mathfrak{n}_+$ (respectively $\mathfrak{n}_-$) is the Lie algebra generated by $\{x_a\}_{a\in\Gamma}$ (respectively $\{y_a\}_{a\in\Gamma}$) subject to the relations XX (respectively YY). The algebra $\mathfrak{b}'_+$ (respectively $\mathfrak{b}'_-$) is the Lie algebra generated by $\{x_a, h_a\}_{a\in\Gamma}$ (respectively $\{y_a, h_a\}_{a\in\Gamma}$) subject to the relations XX, HH, and HX (respectively YY, HH, and HY). We note that $\mathfrak{b}'_+ = \mathfrak{h}' + \mathfrak{n}_+$ and $\mathfrak{b}'_- = \mathfrak{h}' + \mathfrak{n}_-$. The algebra $\mathfrak{g}'$ is the Lie algebra generated by $\{x_a, y_a, h_a\}_{a\in\Gamma}$ subject to all of the relations above. When $\mathfrak{g}$ is finite dimensional (and consequently semisimple), the algebras $\mathfrak{h}$, $\mathfrak{b}_\pm$, and $\mathfrak{g}$ are equal to their derived counterparts. In this dissertation, we are almost always concerned only with the derived subalgebras and $\mathfrak{n}_\pm$.

Let $\mathcal{V}$ be any vector space. We say an operator $T : \mathcal{V} \to \mathcal{V}$ is *square nilpotent* if $T^2 = 0$. Consider actions on $\mathcal{V}$ of the generators x_a, y_a, and h_a that are respectively given by operators X_a, Y_a, and H_a in $\mathrm{End}(\mathcal{V})$ for all $a \in \Gamma$. If the X_a (respectively Y_a) are square nilpotent for all $a \in \Gamma$, we say their actions are collectively *X-square* (respectively *Y-square*) *nilpotent*. An $\mathfrak{h}'$*-weight basis* of $\mathcal{V}$ is a basis $\mathcal{B}$ of $\mathcal{V}$ that simultaneously diagonalizes the operators $\{H_a\}_{a\in\Gamma}$. A *weight function* on $\mathcal{B}$ is a Γ-set of $\mathbb{C}$-valued functions on $\mathcal{B}$. Here the $\mathfrak{h}'$-*weight* of $\{H_a\}_{a\in\Gamma}$ is the weight function $\{\xi_a\}_{a\in\Gamma}$ satisfying $H_a.v = \xi_a(v).v$ for every $a \in \Gamma$ and every $v \in \mathcal{B}$. The *eigenvalue set* for $\mathcal{B}$ is $\mathcal{E}_{\mathfrak{h}'} := \{\xi_a(v) \mid a \in \Gamma, v \in \mathcal{B}\}$.

We recall a basic algebraic fact: Let $k, l \geq 1$ and let $\mathfrak{L}$ be the Lie algebra defined by generators $g_1, \ldots, g_k$ subject to relations $r_1, \ldots, r_l$ in the Lie bracket $[\cdot,\cdot]$ in $\mathfrak{L}$. Let $G_1, \ldots, G_k$ be elements of $\mathrm{End}(\mathcal{V})$. Let $g_i \mapsto G_i$ for $1 \leq i \leq k$ be a bijection. Suppose under this bijection the operators $G_1, \ldots, G_k$ also satisfy the relations $r_1, \ldots, r_l$, now in the commutator $[\cdot,\cdot]$ in $\mathrm{End}(\mathcal{V})$. Then the bijection induces a Lie algebra homomorphism from $\mathfrak{L}$ to the subalgebra of $\mathfrak{gl}(\mathcal{V})$ generated by $G_1, \ldots, G_k$. Hence $\mathcal{V}$ is a representation of $\mathfrak{L}$. Our usage of the terminology above and this algebraic fact will leave the actions of x_a, y_a, and h_a implicit, and so we will refer only to the operators X_a, Y_a, and H_a in $\mathrm{End}(\mathcal{V})$.

2.3 Representations of Lie algebras built from colored posets

We return to our standard context that is established by the fixed locally finite poset P colored with the fixed finite simple graph Γ by κ, and having lattice of splits $\mathcal{FI}(P)$. Let V be the free complex vector space on $\mathcal{FI}(P)$. For each split (F, I), denote the corresponding vector in V by $\langle F, I\rangle$. Every mention of diagonal operators is made with respect to the basis $\{\langle F, I\rangle\}_{(F,I)\in\mathcal{FI}(P)}$ for V. Let $a \in \Gamma$ and let (F, I) be a split in $\mathcal{FI}(P)$. To guarantee the sums in the following two definitions are finite, temporarily assume all antichains in P are finite. Define the *color raising operator* X_a by $X_a.\langle F, I\rangle := \sum\langle F - x, I + x\rangle$; here the sum is taken over all elements x of color a that are minimal in F. Dually, define the *color lowering operator* Y_a by $Y_a.\langle F, I\rangle := \sum\langle F + y, I - y\rangle$; here the sum is taken over all elements y of color a maximal in I. Whenever we create such operators, it will be clear that these sums are finite. Linearly extend X_a and Y_a to all of V. For $a \in \Gamma$, the action of X_a (respectively Y_a) on a basis vector $\langle F, I\rangle$ can be viewed as summing over all ways to move up (respectively down) in $\mathcal{FI}(P)$ from (F, I) by an edge colored a. We extend our standard context to include V (usually implicitly) and the $\{X_a, Y_a\}_{a\in\Gamma}$.

We say $\mathcal{FI}(P)$ *carries a representation of* $\mathfrak{n}_+$ (respectively $\mathfrak{n}_-$) if the $\{X_a\}_{a\in\Gamma}$ (respectively $\{Y_a\}_{a\in\Gamma}$) satisfy XX (respectively YY) with respect to the commutator $[\cdot,\cdot]$ in $\mathrm{End}(V)$. We say $\mathcal{FI}(P)$ *carries a representation of* $\mathfrak{b}'_+$ (respectively $\mathfrak{b}'_-$) if there exists a Γ-set of diagonal operators $\{H_a\}_{a\in\Gamma}$ on V such that $\{X_a, H_a\}_{a\in\Gamma}$ (respectively $\{Y_a, H_a\}_{a\in\Gamma}$) satisfy XX and HX (respectively YY and HY). The lattice of splits in Figure 2.2 carries a representation of $\mathfrak{b}'_+$, where $\mathfrak{g}$ is the algebra of finite type D_5. Finally, we say $\mathcal{FI}(P)$ *carries a representation of* $\mathfrak{g}'$ if there exists a Γ-set of diagonal operators $\{H_a\}_{a\in\Gamma}$ on V such that $\{X_a, Y_a, H_a\}_{a\in\Gamma}$ satisfy

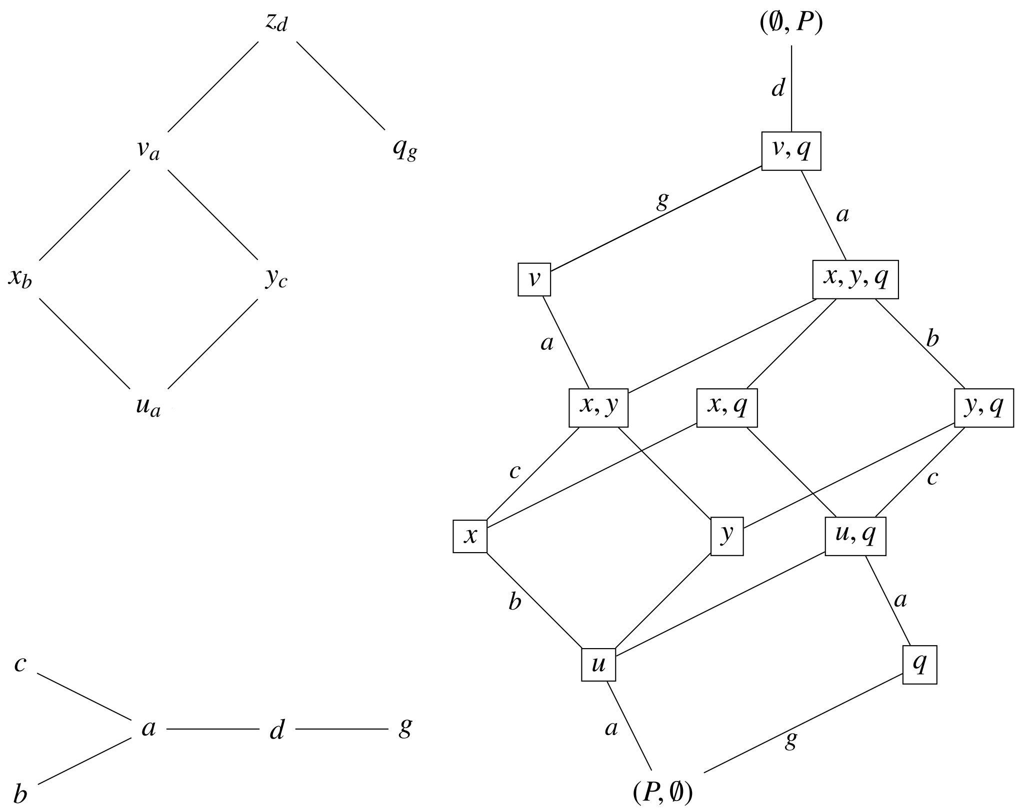


Figure 2.2: Clockwise from bottom left: Simple graph Γ of finite type D_5, poset P colored by Γ, and its edge-colored lattice $\mathcal{FI}(P)$ of filter-ideal splits. Subscripts in P indicate element colors. Splits (F, I) with boxes are described by the maximal elements of the ideal I. Parallel edges in $\mathcal{FI}(P)$ have the same color.

XX, YY, HX, HY, and XY. The lattice of splits in Figure 2.1 carries a representation of $\mathfrak{g}'$, where $\mathfrak{g}$ is the algebra of affine type $\tilde{E}_6$. Note that any diagonal operators $\{H_a\}_{a\in\Gamma}$ automatically satisfy the relation HH. If $\mathcal{FI}(P)$ carries a representation of $\mathfrak{b}'_+$, $\mathfrak{b}'_-$, or $\mathfrak{g}'$, then $\{\langle F, I\rangle\}_{(F,I)\in\mathcal{FI}(P)}$ is always taken to be the $\mathfrak{h}'$-weight basis for V. The minuscule representations built from colored minuscule posets and the representations built from full heaps mentioned in Chapter 1 are examples of representations of $\mathfrak{g}'$ carried by $\mathcal{FI}(P)$. These representations are X- and Y-square nilpotent.

We say a representation of $\mathfrak{b}'_+$ (respectively $\mathfrak{b}'_-$) carried by $\mathcal{FI}(P)$ is *upper* (*lower*) *P-minuscule* if it is X-square (Y-square) nilpotent and the actions of the diagonal operators $\{H_a\}_{a\in\Gamma}$ satisfy:

(i) The set $\mathcal{E}_{\mathfrak{h}'}$ of eigenvalues of the $\{H_a\}_{a\in\Gamma}$ is contained in $\{-1, 0, 1, 2, \dots\}$ (respectively contained in $\{\dots, -2, -1, 0, 1\}$),

(ii) For any split (F, I) and any $a \in \Gamma$ we have $H_a.\langle F, I\rangle = -\langle F, I\rangle$ (respectively $H_a.\langle F, I\rangle = +\langle F, I\rangle$) if and only if F (respectively I) has a minimal (maximal) element of color a.

The representation carried by $\mathcal{FI}(P)$ in Figure 2.2 is upper P-minuscule. Here we have $H_a.\langle P, \emptyset\rangle = -\langle P, \emptyset\rangle$, $H_g.\langle P, \emptyset\rangle = -\langle P, \emptyset\rangle$, $H_b.\langle P, \emptyset\rangle = 0$, $H_c.\langle P, \emptyset\rangle = 0$, and $H_d.\langle P, \emptyset\rangle = +2\langle P, \emptyset\rangle$. The actions of the $\{H_e\}_{e\in\Gamma}$ at other splits can be computed by working up through $\mathcal{FI}(P)$ using the relation HX. So one can see $\mathcal{E}_{\mathfrak{h}'} = \{-1, 0, 1, 2\}$.

We say a representation of $\mathfrak{g}'$ carried by $\mathcal{FI}(P)$ is *P-minuscule* if the set $\mathcal{E}_{\mathfrak{h}'}$ of eigenvalues of the $\{H_a\}_{a\in\Gamma}$ is contained in $\{-1, 0, 1\}$. It can be seen that P-minuscule representations of $\mathfrak{g}'$ are X- and Y-square nilpotent. One can confirm that the lattice $\mathcal{FI}(P)$ of splits displayed in Figure 2.1 carries a P-minuscule representation of $\mathfrak{g}'$.

2.4 The general case: Multiply laced definitions

So far the structure Γ above has been a simple graph. Here we introduce more general structures Γ that can be used to specify general Kac–Moody algebras. These structures are combinatorial reformulations of arbitrary generalized Cartan matrices. The simple graphs from the simply laced case will be subsumed into this new framework.

Let Γ be a finite graph with no loops. We say that Γ is a *Dynkin diagram* if for any two distinct nodes $a, b \in \Gamma$ one of the following three possibilities occurs:

(i) There is no edge between a and b.

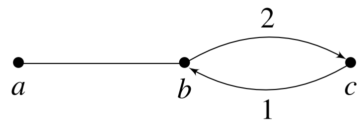
(ii) There is a single undirected and unlabeled edge between a and b.

(iii) There is a pair of oppositely directed edges between a and b, each labeled with a positive integer, such that the product of these two integers is at least 2.

See Figure 2.3 for an example of a Dynkin diagram Γ.

We extend the Section 2.1 definition of the integers θ_{ab} for $a, b \in \Gamma$. We still set $\theta_{aa} := 2$ for all $a \in \Gamma$. Now suppose $a, b \in \Gamma$ are distinct and consider the cases listed above. For Case (i), we still set $\theta_{ab} = \theta_{ba} := 0$ and say a and b are *distant*. For Case (ii), we still set $\theta_{ab} = \theta_{ba} := -1$. For Case (iii), suppose k and l are the positive integers respectively labeling the directed edges from a to b and from b to a. Then we set $\theta_{ab} := -k$ and $\theta_{ba} := -l$. If Case (iii) does not occur for any distinct $a, b \in \Gamma$, then we will still say Γ is *simply laced*. If Case (iii) does occur for some distinct $a, b \in \Gamma$, then we will say Γ is *multiply laced*. In the *general case* we will consider a poset that has been colored by any Dynkin diagram.

Suppose $a, b \in \Gamma$ are distinct nodes satisfying $\theta_{ab} = -k$ and $\theta_{ba} = -l$ for some integers $k, l \geq 1$; that is, we are in Case (ii) or Case (iii). Then we still say a and b are *adjacent* and write $a \sim b$. Now we will also say



$$\begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \end{array}$$

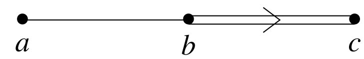


Figure 2.3: From left-to-right: A Dynkin diagram Γ, a generalized Cartan matrix for Γ (using the transpose convention of [Ste]), and the Dynkin diagram of finite type B_3 traditionally used for Γ.

that a is *k-adjacent to b* and b is *l-adjacent to a*. So in Figure 2.3 the color b is 2-adjacent to the color c and c is 1-adjacent to b.

These definitions describe the combinatorial information in a generalized Cartan matrix while avoiding the specification of a total ordering on the nodes of Γ. Once a total ordering of Γ is fixed, the corresponding generalized Cartan matrix is $[\theta_{ab}]_{a,b\in\Gamma}$. Conversely, given a generalized Cartan matrix, one may use its entries to build the corresponding Dynkin diagram satisfying the rules specified above. *A priori* we do not assume that the generalized Cartan matrices associated to our Dynkin diagrams have any special properties, such as being symmetrizable. However, *a posteriori* it can be seen that only certain values of the integers $\{\theta_{ab}\}_{a,b\in\Gamma}$ can arise in the generalized Cartan matrices associated to the Dynkin diagrams that can be used to color the posets that interest us the most. See Corollary 7.5.4.

For a Dynkin diagram Γ, the Lie algebras $\mathfrak{n}_\pm$ and $\mathfrak{b}'_\pm$ and $\mathfrak{g}'$ are now generated by subsets of the symbols $\{x_a, y_a, h_a\}_{a\in\Gamma}$ subject to unions of sets of *general defining relations*:

$$\begin{aligned}
&\text{(XX):} && \underbrace{[x_a, [\dots, [x_a}_{1-\theta_{ba}\text{ times}}, x_b]\dots]] = 0 \text{ for all } a, b \in \Gamma \text{ such that } a \neq b,\\
&\text{(YY):} && \underbrace{[y_a, [\dots, [y_a}_{1-\theta_{ba}\text{ times}}, y_b]\dots]] = 0 \text{ for all } a, b \in \Gamma \text{ such that } a \neq b,\\
&\text{(HH):} && [h_b, h_a] = 0 \text{ for all } a, b \in \Gamma,\\
&\text{(HX):} && [h_b, x_a] = \theta_{ab} x_a \text{ for all } a, b \in \Gamma,\\
&\text{(HY):} && [h_b, y_a] = -\theta_{ab} y_a \text{ for all } a, b \in \Gamma,\\
&\text{(XY):} && [x_a, y_b] = \delta_{ab} h_a \text{ for all } a, b \in \Gamma,
\end{aligned}$$

The algebras $\mathfrak{n}_\pm$, $\mathfrak{b}'_\pm$, and $\mathfrak{g}'$ are defined as in Section 2.2, now using the general defining relations. We still say that $\mathcal{FI}(P)$ *carries a representation* of one of the subalgebras $\mathfrak{n}_\pm$, $\mathfrak{b}'_\pm$, or $\mathfrak{g}'$ with the same definitions as in Section 2.3, now using the general defining relations.

Some of poset coloring properties listed in Section 2.1 will need to be extended as well. Table 2.4 indexes the locations of the updated properties.

Property:	**Abbreviated definition:**	**Location:**	**Extends:**
I3NE	Interval of 3 Neighbors w/ Equal top and bottom colors, middle color k-adjacent to top color for some $k \geq 2$	Lemma 3.3.3	I3ND
I2∨1A	consecutive color Intervals contain 2 or 1 Adjacent colors	Prop. 4.4.2	I2A
MxkSB	MxFGA holds and upper adjacency Sum is Bounded by k	Section 4.6	MxkGA
MnkSB	MnFLA holds and lower adjacency Sum is Bounded by k	Section 4.6	MnkLA

Table 2.4: Extended coloring property abbreviations and locations of definitions

2.5 Extensions of characterizations from the simply laced to the general case

Chapters 3, 4, and 5 are structured as follows: For each section handling the simply laced case in the first half of the chapter, there is a section extending this material to the general case in the second half of the chapter. Result numbers can be compared directly: To extend a result in the simply laced case to the corresponding result in the general case, replace the section number for the simply laced case with the corresponding section number for the general case in the middle of its three-tier number. For example, Proposition 3.1.1 is extended by Proposition 3.3.1.

These extensions are mostly routine. Many proofs can still be applied to the general case verbatim or without much change when using the extended coloring properties and defining relations. When we do rewrite a proof (or a section of a proof), it is usually because the calculations have changed when the extension is made. The most significant change in development happens when extending the simply laced Section 3.1 to the general Section 3.3. There are only two relations XX specifying $\mathfrak{n}_{\pm}$ in the simply laced case, while there are potentially many such relations in the general case. When extending Theorem 3.1.4 to Theorem 3.3.4, a new proof that involves more than just updated calculations is needed. In later proofs, only the calculations change when extending from the simply laced to the general case.

CHAPTER 3

Representations of Borel derived subalgebras

This chapter establishes the earliest results for representations of $\mathfrak{n}_\pm$ and $\mathfrak{b}'_\pm$ built from colored posets. Proposition 3.1.1 contains conditions that are often assumed or derived throughout the rest of the dissertation; these form the foundation for the "minuscule" settings for $\mathfrak{b}'_+$ and $\mathfrak{g}'$ in Chapter 5. After obtaining a characterization of square nilpotent representations of $\mathfrak{n}_\pm$ built from colored posets in Theorem 3.1.4, we extend this result to representations of $\mathfrak{b}'_\pm$ in Theorem 3.2.8.

3.1 Square nilpotent representations of $\mathfrak{n}_+$ and $\mathfrak{n}_-$

We establish our earliest equivalences between sets of coloring properties and sets of algebraic conditions. Theorem 3.1.4 summarizes this section by listing three coloring properties for P that are necessary and sufficient for the specified $\{X_a\}_{a\in\Gamma}$ actions to generate an X-square nilpotent representation of $\mathfrak{n}_+$ that is carried by $\mathcal{FI}(P)$.

Proposition 3.1.1. *The following are equivalent:*

(i) The color raising operators $\{X_a\}_{a\in\Gamma}$ are X-square nilpotent.

(ii) The following two properties are satisfied by P:

(EC): Elements with equal colors are comparable, and

(ND): Neighbors have different colors.

(iii) The color lowering operators $\{Y_a\}_{a\in\Gamma}$ are Y-square nilpotent.

For all $a \in \Gamma$ and every split (F, I), the property EC implies that the sums defining $X_a.\langle F, I\rangle$ and $Y_a.\langle F, I\rangle$ are either single terms or are zero. From now on when we are creating these operators we will be assuming property EC holds. The properties EC and ND together ensure that no two edges of the same color are incident in the Hasse diagram of $\mathcal{FI}(P)$.

Proof. To prove (i) implies (ii), first suppose that EC fails. Then there is a color $a \in \Gamma$ and incomparable elements $x, y \in P_a$. Let F be the filter generated by x and y. Since $2\langle F - x - y, I + x + y\rangle$ is a term in the expansion of $X_a^2.\langle F, I\rangle$, we have $X_a^2.\langle F, I\rangle \neq 0$. Now suppose ND fails. Then there is a color $a \in \Gamma$ and neighbors $x \to y$, with $x, y \in P_a$. Let F be the filter generated by x. Since $\langle F - x - y, I + x + y\rangle$ is a term in the expansion of $X_a^2.\langle F, I\rangle$, we have $X_a^2.\langle F, I\rangle \neq 0$.

Now suppose (ii) holds and fix $a \in \Gamma$. Let $(F, I) \in \mathcal{FI}(P)$. Suppose $X_a.\langle F, I\rangle = \langle F - x, I + x\rangle$ for some element $x \in P_a$. Let $y \in P_a$; here y is comparable to x by EC. However, note y cannot cover x by ND. Hence y is not minimal in $F - x$. Since $y \in P_a$ was arbitrary, we have $X_a.\langle F - x, I + x\rangle = 0$. So for every color $a \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$ we have $X_a^2.\langle F, I\rangle = 0$. Dualize to obtain the equivalence of (ii) and (iii). □

We get our first defining relation by strengthening ND:

Lemma 3.1.2. *Suppose P satisfies EC and ND. Then the following are equivalent:*

(i) The relation $[X_b, X_a] = 0$ *holds if* $a, b \in \Gamma$ *are distant.*

(ii) The following additional property is satisfied by P:

(NA): Neighbors have adjacent colors.

(iii) The relation $[Y_b, Y_a] = 0$ *holds if* $a, b \in \Gamma$ *are distant.*

Proof. To prove (i) implies (ii), first suppose (ii) fails. Then there exist neighbors $x \to y$ such that either $\kappa(x) = \kappa(y)$ or $\kappa(x) \not\sim \kappa(y)$. By ND we have $\kappa(x) \neq \kappa(y)$, so $\kappa(x) \not\sim \kappa(y)$. Let $a := \kappa(x)$ and $b := \kappa(y)$. Let F be the filter generated by x. Since $\langle F - x - y, I + x + y\rangle$ is a term in the expansion of $X_b X_a.\langle F, I\rangle$, we have $X_b X_a.\langle F, I\rangle \neq 0$. However, $X_a X_b.\langle F, I\rangle = 0$ since the only minimal element in F has color a. Hence $[X_b, X_a].\langle F, I\rangle \neq 0$, and so (i) fails.

Now suppose (ii) holds. Let a and b be distant colors. Let (F, I) be any split and without loss of generality assume $X_b X_a.\langle F, I\rangle \neq 0$. Then there are elements x and y such that $\kappa(x) = a$ and $\kappa(y) = b$ and $X_b X_a.\langle F, I\rangle = \langle F - x - y, I + x + y\rangle$. By NA we know x and y are not neighbors. Thus they are incomparable minimal elements of F. Hence $X_a X_b.\langle F, I\rangle = \langle F - y - x, I + y + x\rangle = X_b X_a.\langle F, I\rangle$, so $[X_b, X_a].\langle F, I\rangle = 0$. Dualize to obtain the equivalence of (ii) and (iii). □

We get our second defining relation by introducing a special case of the future key property I2A:

Lemma 3.1.3. *Suppose P satisfies EC and ND. Then the following are equivalent:*

(i) The relation $[X_a, [X_a, X_b]] = 0$ *holds for all* $a, b \in \Gamma$.

(ii) The following additional property is satisfied by P:

(I3ND): If three successive neighbors $x \to y \to z$ form an interval in P, then x and z have different colors.

(iii) The relation $[Y_a, [Y_a, Y_b]] = 0$ holds for all $a, b \in \Gamma$.

Proof. For all $a, b \in \Gamma$, note $[X_a, [X_a, X_b]] = X_a^2 X_b - 2X_a X_b X_a + X_b X_a^2$. By Proposition 3.1.1 the first and last terms vanish when acting on any split. Thus the relation $[X_a, [X_a, X_b]] = 0$ holds if and only if $X_a X_b X_a = 0$ holds.

Suppose (i) holds, so that for all $a, b \in \Gamma$ we have $X_a X_b X_a = 0$. Suppose three successive neighbors $x \to y \to z$ form an interval in P. Define $a := \kappa(x)$ and $b := \kappa(y)$ and $c := \kappa(z)$. Let F be the filter generated by x. Note that $X_c X_b X_a.\langle F, I\rangle = \langle F - x - y - z, I + x + y + z\rangle \neq 0$. Hence we have $c \neq a$, so I3ND holds.

Now suppose (ii) holds. Let $a, b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$ be such that $X_b X_a.\langle F, I\rangle \neq 0$. Then there are elements x and y with $\kappa(x) = a$ and $\kappa(y) = b$ such that $X_b X_a.\langle F, I\rangle = \langle F - x - y, I + x + y\rangle$. Either x and y are incomparable or $x \to y$. Suppose x and y are incomparable. Every element of color a in $F - x - y$ must be greater than x by EC, but none may cover x by ND. Hence $X_a X_b X_a.\langle F, I\rangle = 0$. Now suppose $x \to y$. Suppose $z \in F - x - y$ has color a. By I3ND the set of elements $\{x, y, z\}$ is not an interval. Thus z is not minimal in $F - x - y$, and so again $X_a X_b X_a.\langle F, I\rangle = 0$. Dualize to obtain the equivalence of (ii) and (iii). □

Since NA implies ND, we can combine the three results above to produce:

Theorem 3.1.4. *The following are equivalent:*

(i) The lattice $\mathcal{FI}(P)$ carries an X-square nilpotent representation of $\mathfrak{n}_+$.

(ii) The properties EC, NA, and I3ND are satisfied by P.

(iii) The lattice $\mathcal{FI}(P)$ carries a Y-square nilpotent representation of $\mathfrak{n}_-$.

We conclude this section with a combinatorial property to characterize when the commutator bracket of color raising or lowering operators with adjacent colors is nonzero. We will see at the end of Section 8.3 that "Γ-colored d-complete" and "Γ-colored minuscule" posets (see Section 6.1) satisfy this property.

Proposition 3.1.5. *Suppose P satisfies EC. Then the following are equivalent:*

(i) If $a, b \in \Gamma$ are adjacent, then there is some split (F, I) for which $[X_b, X_a].\langle F, I\rangle \neq 0$.

(ii) The following additional property is satisfied by P:

(AN): Whenever $a, b \in \Gamma$ are adjacent, there exist neighbors $x, y \in P$ with $\kappa(x) = a$ and $\kappa(y) = b$.

(iii) If $a, b \in \Gamma$ are adjacent, then there is some split (F, I) for which $[Y_b, Y_a].\langle F, I\rangle \neq 0$.

Proof. First suppose that AN fails. Then there exist colors $a, b \in \Gamma$ with $a \sim b$ and such that there are no neighbors in P with colors a and b. Let $(F, I) \in \mathcal{FI}(P)$. If $X_b X_a.\langle F, I\rangle = 0$ and $X_a X_b.\langle F, I\rangle = 0$, then we are done. Otherwise, without loss of generality suppose that $X_b X_a.\langle F, I\rangle \neq 0$. By EC, the result of this action is a single term. Thus there are elements $x, y \in P$ with $\kappa(x) = a$ and $\kappa(y) = b$ such that $X_b X_a.\langle F, I\rangle = \langle F - x - y, I + x + y\rangle$. By assumption we know that x and y are not neighbors in P, so they must be incomparable. Thus $X_a X_b.\langle F, I\rangle = \langle F - x - y, I + x + y\rangle$ as well, and so $[X_b, X_a].\langle F, I\rangle = 0$.

Now suppose that AN holds. Suppose $a, b \in \Gamma$ are adjacent. By AN we know there exist some neighbors $x \to y$ with $\kappa(x) = a$ and $\kappa(y) = b$. Let F be the principal filter generated by x. By EC the action $X_b X_a.\langle F, I\rangle$ is a single term, so we have $X_b X_a.\langle F, I\rangle = \langle F - x - y, I + x + y\rangle$. We also have $X_a X_b.\langle F, I\rangle = 0$ since x is the only minimal element of F and $\kappa(x) = a$. Hence $[X_b, X_a].\langle F, I\rangle \neq 0$. Dualize to obtain the equivalence of (ii) and (iii). $\square$

3.2 Square nilpotent representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$

We extend our representations of $\mathfrak{n}_+$ (respectively $\mathfrak{n}_-$) to $\mathfrak{b}'_+$ (respectively $\mathfrak{b}'_-$) by constructing diagonal operators $\{H_a\}_{a\in\Gamma}$ on V that satisfy HX and HY. Locally, if two basis vectors are connected by an edge in $\mathcal{FI}(P)$, the relations HX (or HY) indicate how to relate the $\mathfrak{h}'$-weights of the two vectors:

Lemma 3.2.1. *Suppose P satisfies EC. Let $\{H_a\}_{a\in\Gamma}$ be a Γ-set of diagonal operators with $\mathfrak{h}'$-weight $\{\xi_a\}_{a\in\Gamma}$.*

- *(a) The operators $\{X_a, H_a\}_{a\in\Gamma}$ satisfy the relations HX if and only if for every $b \in \Gamma$, split $(F, I) \in \mathcal{FI}(P)$, and minimal element $x \in F$, we have $\xi_b(F - x, I + x) - \xi_b(F, I) = \theta_{\kappa(x),b}$.*
- *(b) The relations HX are satisfied if and only if the relations HY are satisfied.*
- *(c) The operators $\{Y_a, H_a\}_{a\in\Gamma}$ satisfy the relations HY if and only if for every $b \in \Gamma$, split $(F, I) \in \mathcal{FI}(P)$, and maximal element $y \in I$, we have $\xi_b(F + y, I - y) - \xi_b(F, I) = -\theta_{\kappa(y),b}$.*

Proof. Let $a, b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$. For (a), note that both sides of the requirement $(H_b X_a - X_a H_b).\langle F, I\rangle = \theta_{ab} X_a.\langle F, I\rangle$ vanish if F does not have a minimal element of color a. Let x be minimal in F of color a. By EC we have $X_a.\langle F, I\rangle = \langle F - x, I + x\rangle$. Then $(H_b X_a - X_a H_b).\langle F, I\rangle = (\xi_b(F - x, I + x) - \xi_b(F, I)) X_a.\langle F, I\rangle$, and so the equivalence in (a) follows. Dualize to obtain (c). Part (b) holds since any edge in $\mathcal{FI}(P)$ can be dually viewed both as transferring a minimal element of a filter to an ideal and as transferring a maximal element of an ideal to a filter. $\square$

We will need to compare the $\mathfrak{h}'$-weights of two splits at a distance. Partition $\mathcal{FI}(P)$ into *components* as follows: Two splits (F, I) and (F', I') are in the same component if there is a path from (F, I) to (F', I') in the Hasse diagram of $\mathcal{FI}(P)$ containing finitely many edges. In this section, we work on one component at a time. An example of a poset P for which $\mathcal{FI}(P)$ has more than one component is $P := \mathbb{Z}$. Here there will be three components: The component $\{(\emptyset, \mathbb{Z})\}$, the component $\{(\mathbb{Z}, \emptyset)\}$, and the component containing every other split. Using the finiteness of the set of colors, we get the following:

Proposition 3.2.2. *Suppose P satisfies EC. Then the number of components of $\mathcal{FI}(P)$ is finite. When P is finite, there is only one component.*

Proof. Let $(F, I) \in \mathcal{FI}(P)$. For each $a \in \Gamma$ we could have $P_a \cap I = \emptyset$ or $P_a \cap I = P_a$ or $\emptyset \neq P_a \cap I \neq P_a$. Record this outcome for each color and call the resulting Γ-set of outcomes the *color level* of (F, I). So the split (F, I) can have one of $3^{|\Gamma|}$ possible color levels.

Let (F, I) and (F', I') be splits that have the same color level. We show they are in the same component of $\mathcal{FI}(P)$. Note that

$$I - I' = \bigcup_{a \in \Gamma} P_a \cap (I - I') \text{ and } I' - I = \bigcup_{a \in \Gamma} P_a \cap (I' - I).$$

Since (F, I) and (F', I') have the same color level and P satisfies EC, we see for all $a \in \Gamma$ that $P_a \cap (I - I')$ and $P_a \cap (I' - I)$ are finite sets. The unions on each right hand side are disjoint and Γ is finite, so we have

$$|I - I'| = \sum_{a \in \Gamma} |P_a \cap (I - I')| \text{ and } |I' - I| = \sum_{a \in \Gamma} |P_a \cap (I' - I)|.$$

Thus $I - I'$ and $I' - I$ are finite sets. Since a path from (F, I) to (F', I') can be formed by transferring all of the elements of $I - I'$ to a filter and then transferring all of the elements of $I' - I$ to an ideal, we see (F, I) and (F', I') are in the same component of $\mathcal{FI}(P)$. Since there are finitely many color levels and the splits that have the same color level are in the same component of $\mathcal{FI}(P)$, there must be finitely many components. If P is finite, then so are $I - I'$ and $I' - I$ for any $(F, I), (F', I') \in \mathcal{FI}(P)$. Thus there is only one component in this case. □

Here we will only use the second statement of Proposition 3.2.2.

Suppose (F, I) and (F', I') are in the same component of $\mathcal{FI}(P)$. The edges in any path from (F, I) to (F', I') have the net effect of adding every element of $I' - I$ and removing every element of $I - I'$. Fix $b \in \Gamma$ and note that both $P_b \cap (I' - I)$ and $P_b \cap (I - I')$ are finite. So in any path from (F, I) to (F', I'), the cardinality $|P_b \cap (I' - I)|$ counts the net number of edges of color b traversed "going up" and $|P_b \cap (I - I')|$ counts the net number of edges of color b traversed "going down." Set $\Delta_b[(F',I'),(F,I)] := |P_b \cap (I'-I)| - |P_b \cap (I-I')|$. This is the signed net number of edges of color b traversed in any finite path from (F,I) to (F',I'). Note that $\Delta_b[(F',I'),(F,I)] = -\Delta_b[(F,I),(F',I')]$. We are defining the functions $\{\Delta_a\}_{a\in\Gamma}$ only on pairs of splits that are in the same component of $\mathcal{FI}(P)$.

We now present a components-wide version of the equations in Lemma 3.2.1. Let $\{\eta_a\}_{a\in\Gamma}$ be a weight function on $\mathcal{FI}(P)$. We call it a *component weight function* if for all $b \in \Gamma$, whenever (F,I) and (F',I') are in the same component of $\mathcal{FI}(P)$ we have

$$\eta_b(F',I') - \eta_b(F,I) = 2\Delta_b[(F',I'),(F,I)] - \sum_{c\sim b} \Delta_c[(F',I'),(F,I)]. \tag{3.1}$$

We say that a Γ-set of diagonal operators $\{H_a\}_{a\in\Gamma}$ on V are *component diagonal operators* if their $\mathfrak{h}'$-weight $\{\eta_a\}_{a\in\Gamma}$ is a component weight function.

Lemma 3.2.3. *Let $b \in \Gamma$ and let (F,I), (F',I'), and (F'',I'') be splits in the same component of $\mathcal{FI}(P)$.*

(a) We have $\Delta_b[(F'',I''),(F,I)] = \Delta_b[(F'',I''),(F',I')] + \Delta_b[(F',I'),(F,I)]$.

(b) Let $\{\eta_a\}_{a\in\Gamma}$ be a weight function on $\mathcal{FI}(P)$. If Equation (3.1) holds for the pairs of splits $(F,I),(F',I')$ and $(F',I'),(F'',I'')$, then it also holds for the pair $(F,I),(F'',I'')$.

Proof. Let $\mathcal{P}_1$ be a finite path in $\mathcal{FI}(P)$ from (F,I) to (F',I'), and let $\mathcal{P}_2$ be a finite path in $\mathcal{FI}(P)$ from (F',I') to (F'',I''). Let $\mathcal{P}$ be the concatenation of $\mathcal{P}_1$ followed by $\mathcal{P}_2$. The quantities $\Delta_b[(F',I'),(F,I)]$ and $\Delta_b[(F'',I''),(F',I')]$ are the respective signed net number of edges of color b traversed along $\mathcal{P}_1$ and $\mathcal{P}_2$. Thus their sum is the signed net number of edges of color b traversed along $\mathcal{P}$. But this quantity is also $\Delta_b[(F'',I''),(F,I)]$, so we get (a). Then (b) follows from (a) by noting that $\eta_b(F'',I'') - \eta_b(F,I) = \eta_b(F'',I'') - \eta_b(F',I') + \eta_b(F',I') - \eta_b(F,I)$. □

Applying Equation (3.1) to an edge in $\mathcal{FI}(P)$ recovers the equations in Parts (a) and (c) of Lemma 3.2.1: Any edge in $\mathcal{FI}(P)$ has the form $(F,I) \to (F-x, I+x)$ for some split (F,I) and some minimal element x of F. Set $a := \kappa(x)$. Note that for any $d \in \Gamma$ we have $\Delta_d[(F-x,I+x),(F,I)] = \delta_{ad}$. Applying this observation to (3.1) for some $b \in \Gamma$ results in

$$\eta_b(F-x, I+x) - \eta_b(F,I) = \theta_{ab}, \tag{3.2}$$

since $\theta_{ab} = 2\delta_{ab} - \sum_{c\sim b}\delta_{ac}$. This is the equation from Lemma 3.2.1(a). The equation from Lemma

3.2.1(c) can be obtained by using the alternate realization $(F' + x, I' - x) \to (F', I')$ of this edge, where $(F', I') := (F - x, I + x)$. This specialization gives the “necessary” direction and the transitivity from Lemma 3.2.3(b) gives the “sufficient” direction of

Corollary 3.2.4. *A weight function is a component weight function if and only if it satisfies Equation (3.2) along every edge of $\mathcal{FI}(P)$.*

With this result in mind, applying Lemma 3.2.1 shows the operators needed to satisfy HX (or HY) are exactly the component diagonal operators:

Proposition 3.2.5. *Suppose P satisfies EC. Let $\{H_a\}_{a\in\Gamma}$ be diagonal operators with $\mathfrak{h}'$-weight $\{\eta_a\}_{a\in\Gamma}$. Then the following are equivalent:*

(i) The operators $\{X_a, H_a\}_{a\in\Gamma}$ satisfy HX.

(ii) The operators $\{H_a\}_{a\in\Gamma}$ are component diagonal operators.

(iii) The operators $\{Y_a, H_a\}_{a\in\Gamma}$ satisfy HY.

Apply this result to representations:

Corollary 3.2.6. *Suppose P satisfies EC.*

(a) If $\mathcal{FI}(P)$ carries a representation of $\mathfrak{b}'_+$ or of $\mathfrak{b}'_-$, then the diagonal operators $\{H_a\}_{a\in\Gamma}$ giving the actions of $\{h_a\}_{a\in\Gamma}$ are component diagonal operators.

(b) Suppose P additionally satisfies NA and I3ND. If $\{\eta_a\}_{a\in\Gamma}$ is a component weight function, then the corresponding component diagonal operators $\{H_a\}_{a\in\Gamma}$ can be used to extend the representations of Theorem 3.1.4 from $\mathfrak{n}_+$ and $\mathfrak{n}_-$ to X- and Y-square nilpotent representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$.

Now we show a component weight function $\{\eta_a\}_{a\in\Gamma}$ always exists. Fix $b \in \Gamma$, a component C, and any split $(F_0, I_0) \in C$. Let $\eta_b(F_0, I_0)$ be any complex number. Then for $(F, I) \in C$, define

$$\eta_b(F, I) := \eta_b(F_0, I_0) + 2\Delta_b[(F, I), (F_0, I_0)] - \sum_{c\sim b} \Delta_c[(F, I), (F_0, I_0)]. \tag{3.3}$$

For each $b \in \Gamma$, make such choices for all components and then perform this construction.

Lemma 3.2.7. *(a) The $\{\eta_a\}_{a\in\Gamma}$ defined by (3.3) is a component weight function.*

(b) Each η_a in some component weight function $\{\eta_a\}_{a\in\Gamma}$ is uniquely determined by its value on one split for each component.

Proof. Fix $b \in \Gamma$, a component C, and the split (F_0, I_0) chosen for C. Let $(F, I), (F', I') \in C$. Using (3.3) we know Equation (3.1) holds for the pairs $(F, I), (F_0, I_0)$ and $(F', I'), (F_0, I_0)$. Since Equation (3.1) holds also for the pair $(F_0, I_0), (F', I')$, Lemma 3.2.3(b) shows it holds for $(F, I), (F', I')$. Thus we get (a). We get (b) by noting that each η_a in a given $\{\eta_a\}_{a\in\Gamma}$ must satisfy (3.3) once its values are specified on one split from each component. □

Using the existence obtained in Lemma 3.2.7(a), we can extend Theorem 3.1.4:

Theorem 3.2.8. *The following are equivalent:*

(i) The lattice $\mathcal{F}\mathcal{I}(P)$ carries an X-square nilpotent representation of $\mathfrak{n}_+$.

(ii) The lattice $\mathcal{F}\mathcal{I}(P)$ carries an X-square nilpotent representation of $\mathfrak{b}'_+$.

(iii) The properties EC, NA, and I3ND are satisfied by P.

(iv) The lattice $\mathcal{F}\mathcal{I}(P)$ carries a Y-square nilpotent representation of $\mathfrak{b}'_-$.

(v) The lattice $\mathcal{F}\mathcal{I}(P)$ carries a Y-square nilpotent representation of $\mathfrak{n}_-$.

Any operators $\{H_a\}_{a\in\Gamma}$ used to satisfy Part (ii) or Part (iv) are component diagonal operators.

Proof. Using Lemma 3.2.7(a), let $\{\eta_a\}_{a\in\Gamma}$ be any component weight function. Then (iii) implies (ii) by Corollary 3.2.6(b) using $\{\eta_a\}_{a\in\Gamma}$. Also (ii) implies (i) by restricting to the operators $\{X_a\}_{a\in\Gamma}$. And (i) implies (iii) by Theorem 3.1.4. Dualize to obtain the equivalence of (iii), (iv), and (v). The last statement follows from Corollary 3.2.6(a). □

3.3 Square nilpotent representations of $\mathfrak{n}_+$ and $\mathfrak{n}_-$ in the general case

We now extend the results of Chapter 3 to the general case. See Section 2.5 for a description of numbering conventions and a summary of changes between the simply laced and general cases. Here we refer to the definitions for the general case made in Section 2.4. As we generalize it, the statement of our first result does not need to be modified:

Proposition 3.3.1. *The following are equivalent:*

(i) The color raising operators $\{X_a\}_{a\in\Gamma}$ are X-square nilpotent.

(ii) The poset P satisfies EC and ND.

(iii) The color lowering operators $\{Y_a\}_{a\in\Gamma}$ are Y-square nilpotent.

The proof of Proposition 3.1.1 given in Section 3.1 can be applied verbatim since all entities in this result have the same definitions in the simply laced and general cases.

By Proposition 3.3.1, we see that when EC and ND hold we have

$$[X_a, [X_a, [X_a, X_b]]] = 0 \text{ for all } a, b \in \Gamma, \tag{3.4}$$

since X_a^2 will be in every term of the commutator expansion of the left-hand side. So for the operators $\{X_a\}_{a\in\Gamma}$ to satisfy XX for any value of θ_{ba}, we only need to be concerned with the brackets $[X_a, X_b]$ and $[X_a, [X_a, X_b]]$ for every $a, b \in \Gamma$; if $\theta_{ba} \leq -2$, the remaining requirements for XX will hold by applying (3.4).

We reverse the roles of a and b in Lemma 3.1.2 to handle the extended relations XX; the original proof can still be used.

Lemma 3.3.2. *Suppose P satisfies EC and ND. Then the following are equivalent:*

(i) The relation $[X_a, X_b] = 0$ holds if $a, b \in \Gamma$ are distant.

(ii) The poset P satisfies NA.

(iii) The relation $[Y_a, Y_b] = 0$ holds if $a, b \in \Gamma$ are distant.

Lemma 3.1.3 must be modified. This modification introduces the property I3NE, which updates the property I3ND. We note that I3NE is a preliminary version of I2∨1A, which will extend I2A in Section 4.4. Here we need to rewrite most of the proof.

Lemma 3.3.3. *Suppose P satisfies EC and ND. Then the following are equivalent:*

(i) If $[X_a, [X_a, X_b]] \neq 0$ for $a, b \in \Gamma$, then $\theta_{ba} \leq -2$.

(ii) The following additional property is satisfied by P:

> *(I3NE): If three successive neighbors $x \to y \to z$ form an interval in P and x and z have equal colors, then $\kappa(y)$ is k-adjacent to $\kappa(x) = \kappa(z)$ for some $k \geq 2$.*

(iii) If $[Y_a, [Y_a, Y_b]] \neq 0$ for $a, b \in \Gamma$, then $\theta_{ba} \leq -2$.

Proof. For $a, b \in \Gamma$: As before, by Proposition 3.3.1 the relation $[X_a, [X_a, X_b]] = 0$ holds if and only if $X_a X_b X_a = 0$ holds.

Suppose that (i) holds. Suppose three successive neighbors $x \to y \to z$ form an interval in P and $a := \kappa(x) = \kappa(z)$. Let $b := \kappa(y)$. Let F be the filter generated by x and let $I := P - F$. Then $X_a X_b X_a.\langle F, I\rangle = \langle F - x - y - z, I + x + y + z\rangle \neq 0$. By (i) we know $\theta_{ba} \leq -2$, so b is k-adjacent to a for some $k \geq 2$.

Now suppose that (ii) holds. Let $a, b \in \Gamma$ and suppose that $[X_a, [X_a, X_b]] \neq 0$. By the first paragraph of this proof we know $X_a X_b X_a \neq 0$. Then there is a split $(F, I) \in \mathcal{FI}(P)$ and elements $x, y, z \in P$ such that

$X_aX_bX_a.\langle F, I\rangle = \langle F - x - y - z, I + x + y + z\rangle$. Here we have $\kappa(x) = \kappa(z) = a$ and $\kappa(y) = b$. By EC we have $x < z$. By ND we know that z does not cover x. Let u be any element in the open interval (x, z). We know z is minimal in $F - x - y$; from this we see that $u = y$. Hence $x \to y \to z$ is an interval. From I3NE we see that b is k-adjacent to a for some $k \geq 2$, so $\theta_{ba} \leq -2$. Dualize to obtain the equivalence of (ii) and (iii). □

To extend the statement of Theorem 3.1.4, we only need to replace the property I3ND by the updated property I3NE. We include a new proof, which is needed due to the more complicated extended relations XX.

Theorem 3.3.4. *The following are equivalent:*

(i) The lattice $\mathcal{FI}(P)$ carries an X-square nilpotent representation of $\mathfrak{n}_+$.

(ii) The properties EC, NA, and I3NE are satisfied by P.

(iii) The lattice $\mathcal{FI}(P)$ carries a Y-square nilpotent representation of $\mathfrak{n}_-$.

Proof. Suppose (i) holds. Proposition 3.3.1 shows that EC and ND hold. Note for $a, b \in \Gamma$ that we have $\theta_{ba} = 0$ if and only if a and b are distant. Thus the relations XX and Lemma 3.3.2 show that P satisfies NA. If for some $a, b \in \Gamma$ we have $[X_a, [X_a, X_b]] \neq 0$, then the relation XX implies $\theta_{ba} \leq -2$. So Lemma 3.3.3 shows that P satisfies I3NE.

Now suppose (ii) holds. Since NA implies ND, we see P satisfies EC and ND. By Proposition 3.3.1 we know the actions of the $\{X_a\}_{a\in\Gamma}$ are X-square nilpotent. Let $a, b \in \Gamma$. Suppose $\theta_{ba} = 0$. Since this is equivalent to the condition that a and b are distant, Lemma 3.3.2 shows that $[X_a, X_b] = 0$. Now suppose $\theta_{ba} = -1$. Then Lemma 3.3.3 shows that $[X_a, [X_a, X_b]] = 0$. Also, by (3.4) we know that $[X_a, [X_a, [X_a, X_b]]] = 0$ for all $a, b \in \Gamma$. Hence the relations XX hold if $\theta_{ba} \leq -2$ as well, so (i) holds. Dualize to obtain the equivalence of (ii) and (iii). □

So even though the relations XX now possibly contain many more requirements than they did previously, we only needed a simple alteration of I3ND in order to extend the characterization of the X-square nilpotent representations of $\mathfrak{n}_\pm$ in Theorem 3.1.4 to the general case.

The extension of Proposition 3.1.5 has the same statement and proof as before:

Proposition 3.3.5. *Suppose P satisfies EC. Then the following are equivalent:*

(i) If $a, b \in \Gamma$ are adjacent, then there is some split (F, I) for which $[X_b, X_a].\langle F, I\rangle \neq 0$.

(ii) The property AN is satisfied by P.

(iii) If $a, b \in \Gamma$ are adjacent, then there is some split (F, I) for which $[Y_b, Y_a].\langle F, I\rangle \neq 0$.

3.4 Square nilpotent representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ in the general case

Section 3.2 introduced several accounting tools for tracking the actions of the simple coroots in representations of $\mathfrak{b}'_\pm$ built from colored posets. These tools are not affected by the updated definition of θ_{ab} for $a, b \in \Gamma$. Our computations in Section 3.2 depended only upon the colored edge structure of the distributive lattice $\mathcal{FI}(P)$ and would hold for *any* complex valued matrix $[\theta_{ab}]_{a,b\in\Gamma}$. Lemma 3.2.1 has the same statement and proof as before:

Lemma 3.4.1. *Suppose P satisfies EC. Let $\{H_a\}_{a\in\Gamma}$ be a Γ-set of diagonal operators with $\mathfrak{h}'$-weight $\{\xi_a\}_{a\in\Gamma}$.*

- *(a) The operators $\{X_a, H_a\}_{a\in\Gamma}$ satisfy the relations HX if and only if for every $b \in \Gamma$, split $(F, I) \in \mathcal{FI}(P)$, and minimal element $x \in F$, we have $\xi_b(F - x, I + x) - \xi_b(F, I) = \theta_{\kappa(x),b}$.*
- *(b) The relations HX are satisfied if and only if the relations HY are satisfied.*
- *(c) The operators $\{Y_a, H_a\}_{a\in\Gamma}$ satisfy the relations HY if and only if for every $b \in \Gamma$, split $(F, I) \in \mathcal{FI}(P)$, and maximal element $y \in I$, we have $\xi_b(F + y, I - y) - \xi_b(F, I) = -\theta_{\kappa(y),b}$.*

Components of $\mathcal{FI}(P)$ and the Γ-set of functions $\{\Delta_a\}_{a\in\Gamma}$ have the same definitions as in Section 3.2. However, Equation (3.1) took advantage of having $\theta_{bc} = -1$ when b and c are adjacent colors. Let $\{\eta_a\}_{a\in\Gamma}$ be a weight function on $\mathcal{FI}(P)$. Generalizing Equation (3.1), we define $\{\eta_a\}_{a\in\Gamma}$ to be a *component weight function* if for all $b \in \Gamma$, whenever (F, I) and (F', I') are in the same component of $\mathcal{FI}(P)$ we have

$$\eta_b(F', I') - \eta_b(F, I) = 2\Delta_b[(F', I'), (F, I)] + \sum_{c\sim b} \theta_{cb}\Delta_c[(F', I'), (F, I)]. \tag{3.5}$$

In practice, we will use the following equivalent form of Equation (3.5):

$$\eta_b(F', I') - \eta_b(F, I) = \sum_{c\in\Gamma} \theta_{cb}\Delta_c[(F', I'), (F, I)]. \tag{3.6}$$

This version is obtained by combining the terms on the right-hand side of (3.5) using $\theta_{bb} = 2$ and $\theta_{cb} = 0$ when $c \not\sim b$.

In the extension of Lemma 3.2.3 we only need to update the equation number:

Lemma 3.4.3. *Let $b \in \Gamma$ and let (F, I), (F', I'), and (F'', I'') be splits in the same component of $\mathcal{FI}(P)$.*

- *(a) We have $\Delta_b[(F'', I''), (F, I)] = \Delta_b[(F'', I''), (F', I')] + \Delta_b[(F', I'), (F, I)]$.*
- *(b) Let $\{\eta_a\}_{a\in\Gamma}$ be a weight function on $\mathcal{FI}(P)$. If Equation (3.6) holds for the pairs of splits $(F, I), (F', I')$ and $(F', I'), (F'', I'')$, then it also holds for the pair $(F, I), (F'', I'')$.*

The original proof of Lemma 3.2.3 can be applied verbatim.

The argument given following Lemma 3.2.3 that produced Equation 3.2 still works, once the following modifications are made: Now refer to Equation (3.6), and remove the phrase "since $\theta_{ab} = 2\delta_{ab} - \sum_{c\sim b} \delta_{ac}$." This phrase is not needed since the right-hand side of Equation (3.6) becomes θ_{ab} directly. For the reader's convenience, we restate the unchanged Equation (3.2):

$$\eta_b(F - x, I + x) - \eta_b(F, I) = \theta_{ab}. \tag{3.2}$$

Here b is an arbitrary color, the element x is minimal in F, and $a := \kappa(x)$. So Corollary 3.2.4 still holds verbatim:

Corollary 3.4.4. *A weight function is a component weight function if and only if it satisfies Equation (3.2) along every edge of $\mathcal{FI}(P)$.*

This corollary and Lemma 3.4.1 will reproduce Proposition 3.2.5 verbatim:

Proposition 3.4.5. *Suppose P satisfies EC. Let $\{H_a\}_{a\in\Gamma}$ be diagonal operators with $\mathfrak{h}'$-weight $\{\eta_a\}_{a\in\Gamma}$. Then the following are equivalent:*

(i) The operators $\{X_a, H_a\}_{a\in\Gamma}$ satisfy HX.

(ii) The operators $\{H_a\}_{a\in\Gamma}$ are component diagonal operators.

(iii) The operators $\{Y_a, H_a\}_{a\in\Gamma}$ satisfy HY.

Now apply this Proposition 3.4.5 to representations and use Theorem 3.3.4 to extend Corollary 3.2.6; the only change is the updating of property I3ND to property I3NE:

Corollary 3.4.6. *Suppose P satisfies EC.*

(a) If $\mathcal{FI}(P)$ carries a representation of $\mathfrak{b}'_+$ or of $\mathfrak{b}'_-$, then the diagonal operators $\{H_a\}_{a\in\Gamma}$ giving the actions of $\{h_a\}_{a\in\Gamma}$ are component diagonal operators.

(b) Suppose P additionally satisfies NA and I3NE. If $\{\eta_a\}_{a\in\Gamma}$ is a component weight function, then the corresponding component diagonal operators $\{H_a\}_{a\in\Gamma}$ can be used to extend the representations of Theorem 3.3.4 from $\mathfrak{n}_+$ and $\mathfrak{n}_-$ to X- and Y-square nilpotent representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$.

A component weight function can always be constructed as before: Use the same setup and update Equation

(3.3) to the general

$$\eta_b(F, I) := \eta_b(F_0, I_0) + \sum_{c \in \Gamma} \theta_{cb} \Delta_c[(F, I), (F_0, I_0)]. \tag{3.7}$$

In Lemma 3.2.7 we only need to update the equation number:

Lemma 3.4.7. *(a) The $\{\eta_a\}_{a\in\Gamma}$ defined by (3.7) is a component weight function.*

(b) Each η_a in some component weight function $\{\eta_a\}_{a\in\Gamma}$ is uniquely determined by its value on one split for each component.

The proof of Lemma 3.2.7 in Section 3.2 still applies once Equations (3.6) and (3.7) are referenced. We can now extend Theorem 3.2.8 to the general case; again the only change in the statement is to replace property I3ND with I3NE.

Theorem 3.4.8. *The following are equivalent:*

(i) The lattice $\mathcal{FI}(P)$ carries an X-square nilpotent representation of $\mathfrak{n}_+$.

(ii) The lattice $\mathcal{FI}(P)$ carries an X-square nilpotent representation of $\mathfrak{b}'_+$.

(iii) The properties EC, NA, and I3NE are satisfied by P.

(iv) The lattice $\mathcal{FI}(P)$ carries a Y-square nilpotent representation of $\mathfrak{b}'_-$.

(v) The lattice $\mathcal{FI}(P)$ carries a Y-square nilpotent representation of $\mathfrak{n}_-$.

Any operators $\{H_a\}_{a\in\Gamma}$ used to satisfy Part (ii) or Part (iv) are component diagonal operators.

The original proof of Theorem 3.2.8 can be applied verbatim.

CHAPTER 4

Combinatorially characterized weights

Here we begin to examine $\mathfrak{h}'$-weights more closely by examining the eigenvalues of component diagonal operators on our basis of splits $\{\langle F, I\rangle\}_{(F,I)\in\mathcal{FI}(P)}$. The minuscule representations of the semisimple Lie algebras act in such a way that for every color $a \in \Gamma$ and any weight basis vector v, the action of h_a on v gives $h_a.v = -v$ or $h_a.v = 0$ or $h_a.v = +v$. The operators and properties we introduce in this chapter will be used in Chapter 5 to obtain this condition in our setting of representations constructed from colored posets. Our main result in this chapter is Proposition 4.2.1; it obtains a combinatorial characterization of the "if" direction in Part (ii) of the definition of an upper P-minuscule representation of $\mathfrak{b}'_+$.

4.1 A combinatorially motivated component weight function

We continue to assume P satisfies EC. If one does not care about the relationship between combinatorial properties and eigenvalues, Theorem 3.2.8 said that a representation of $\mathfrak{n}_+$ carried by $\mathcal{FI}(P)$ can be extended to $\mathfrak{b}'_+$ without requiring coloring properties for P beyond EC, NA, and I3ND: Create a component weight function by choosing for each component of $\mathcal{FI}(P)$ a Γ-set of complex numbers and a split. Then use the corresponding component diagonal operators $\{H_a\}_{a\in\Gamma}$ to extend the action of $\mathfrak{n}_+$. Here we construct a particular weight function $\{\mu_a\}_{a\in\Gamma}$ on $\mathcal{FI}(P)$ whose values are determined by the local structure of P. When P has two new additional properties beyond EC, in Proposition 4.1.2 we use Corollary 3.2.4 to show that it is a component weight function. As we work toward obtaining the attractive (upper) P-minuscule representations of $\mathfrak{g}'$ (and $\mathfrak{b}'_+$), in the next two sections we will obtain relationships between further coloring properties and this $\mathfrak{h}'$-weight.

We prepare to define our new $\mathbb{Z}$-valued weight function $\{\mu_a\}_{a\in\Gamma}$. Fix a color $b \in \Gamma$. To construct $\mu_b : \mathcal{FI}(P) \to \mathbb{Z}$, we introduce auxiliary functions $\upsilon_b : \mathcal{FI}(P) \to \mathbb{N}$ and $\psi_b : \mathcal{FI}(P) \to \mathbb{N}$:

We first define $\upsilon_b : \mathcal{FI}(P) \to \mathbb{N}$ in stages. Let (F, I) be a split. If $P_b \cap I$ does not have a maximal element, then set $\upsilon_b(F, I) := 1$. Now suppose that $P_b \cap I$ has a maximal element y. By EC the element y is unique. We build up a set $\Upsilon_b(F, I) \subseteq I$ from the empty set $\emptyset$. Let $z \in I$. We place z into $\Upsilon_b(F, I)$ if it meets the following three requirements:

(i) The element z is greater than y,

(ii) Its color $c := \kappa(z)$ is adjacent to b, and

(iii) The number of elements greater than y that are in $P_c \cap I$ is finite.

If there is some color $a \sim b$ such that there are infinitely many elements greater than y in $P_a \cap I$, then set $\upsilon_b(F, I) := |\Upsilon_b(F, I)| + 1$. Otherwise set $\upsilon_b(F, I) := |\Upsilon_b(F, I)|$.

Figure 4.1 illustrates the three possible scenarios for computing $\{\upsilon_i\}_{i\in\Gamma}$ on a Γ-colored poset P at a split (F, I). Since there is no maximal element of color d in I, one has $\upsilon_d(F, I) = 1$. Because $g \sim d$, we get $\upsilon_g(F, I) = |\Upsilon_g(F, I)| + 1 = 2$. Lastly, we have $\upsilon_a(F, I) = |\Upsilon_a(F, I)| = 0$.

We also define $\psi_b : \mathcal{FI}(P) \to \mathbb{N}$ in dually analogous stages. Let (F, I) be a split. If $P_b \cap F$ does not have a minimal element, then set $\psi_b(F, I) := 1$. Now suppose that $P_b \cap F$ has a minimal element y. By EC the element y is unique. We build up another set $\Psi_b(F, I) \subseteq F$ from the empty set $\emptyset$. Let $z \in F$. We place z into $\Psi_b(F, I)$ if it meets the following three requirements:

(i) The element z is less than y,

(ii) Its color $c := \kappa(z)$ is adjacent to b, and

(iii) The number of elements less than y that are in $P_c \cap F$ is finite.

If there is some color $a \sim b$ such that there are infinitely many elements less than y in $P_a \cap F$, then set $\psi_b(F, I) := |\Psi_b(F, I)| + 1$. Otherwise set $\psi_b(F, I) := |\Psi_b(F, I)|$.

We can now define the weight function $\{\mu_a\}_{a\in\Gamma}$. Let (F, I) be a split and let $b \in \Gamma$. If $P_b \cap I \neq \emptyset$, then

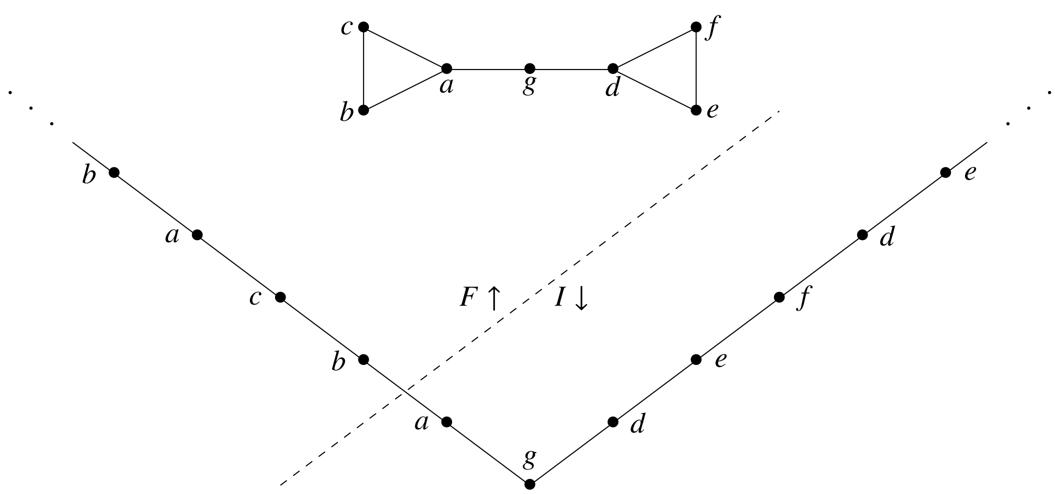


Figure 4.1: Simple graph Γ and poset P colored by Γ. Split (F, I) is used to illustrate computation of $\{\upsilon_i\}_{i\in\Gamma}$.

define $\mu_b(F, I) := 1 - \upsilon_b(F, I)$. If $P_b \cap I = \emptyset$, then define $\mu_b(F, I) := -1 + \psi_b(F, I)$. Finally, define the Γ-set of operators $\{M_a\}_{a\in\Gamma}$ to be the diagonal operators with $\mathfrak{h}'$-weight $\{\mu_a\}_{a\in\Gamma}$. These are the *μ-diagonal operators*.

Foreshadowing Proposition 4.2.1, it is already easy to see that $M_b.\langle F, I\rangle = +\langle F, I\rangle$ if I has a maximal element of color b. Here are some technical comments: When defining $\upsilon_b(F, I)$, we added $+1$ to $|\Upsilon_b(F, I)|$ when there was a color $a \sim b$ such that there were infinitely many elements in $P_a \cap I$ greater than the maximal element of color b. For Sections 4.1–4.2 we could have instead added any complex number to $|\Upsilon_b(F, I)|$. But this number must be a positive integer for the proof of Theorem 5.1.1 to work. The dual comment applies to $\psi_b(F, I)$. The definition of $\{\mu_a\}_{a\in\Gamma}$ is not symmetric with respect to F and I; an alternate construction can be made for the μ-diagonal operators that emphasizes filters instead of ideals. Let $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$. If $P_b \cap F = \emptyset$, then set $\mu'_b(F, I) := 1 - \upsilon_b(F, I)$. If $P_b \cap F \neq \emptyset$, then set $\mu'_b(F, I) := -1 + \psi_b(F, I)$.

Proposition 4.1.1. *Suppose P satisfies EC and the following additional properties:*

(AC): Elements with adjacent colors are comparable, and

(I2A): For every $a \in \Gamma$: The open interval between any two consecutive elements of color a contains exactly two elements whose colors are adjacent to a.

Then $\mu'_b = \mu_b$ for all $b \in \Gamma$.

It is easy to see that this new property I2A implies both ND and I3ND.

Proof. Let $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$. Since κ is surjective, there are three cases to consider: (i) $P_b \cap I = \emptyset$ and $P_b \cap F \neq \emptyset$; (ii) $P_b \cap I \neq \emptyset$ and $P_b \cap F \neq \emptyset$; and (iii) $P_b \cap I \neq \emptyset$ and $P_b \cap F = \emptyset$. For (i) we have $\mu_b(F, I) = -1 + \psi_b(F, I) = \mu'_b(F, I)$. For (iii) we have $\mu_b(F, I) = 1 - \upsilon_b(F, I) = \mu'_b(F, I)$. For Case (ii) we have $\mu_b(F, I) = 1 - \upsilon_b(F, I)$ and $\mu'_b(F, I) = -1 + \psi_b(F, I)$. By local finiteness I has a maximal element y of color b and F has a minimal element z of color b. By EC we know that $y < z$, and these must be consecutive elements of color b. By I2A we know (y, z) has two elements u and v with colors adjacent to b. By AC, any element in $\Psi_b(F, I)$ or $\Upsilon_b(F, I)$ must be in (y, z). Thus we have $\Psi_b(F, I) \subseteq \{u, v\}$ and $\Upsilon_b(F, I) \subseteq \{u, v\}$. Note that u and v must each be in exactly one of the sets $\Psi_b(F, I)$ or $\Upsilon_b(F, I)$. Thus we have $\psi_b(F, I) + \upsilon_b(F, I) = 2$ since $\psi_b(F, I) = |\Psi_b(F, I)|$ and $\upsilon_b(F, I) = |\Upsilon_b(F, I)|$. Hence $\mu_b(F, I) = 1 - \upsilon_b(F, I) = 1 - (2 - \psi_b(F, I)) = -1 + \psi_b(F, I) = \mu'_b(F, I)$. □

We continue to work with $\{\mu_a\}_{a\in\Gamma}$. When AC and I2A hold, it is a component weight function:

Proposition 4.1.2. *Suppose P satisfies EC, AC, and I2A. Then the weight function $\{\mu_a\}_{a\in\Gamma}$ is a component weight function. If P further satisfies NA, then the corresponding operators $\{M_a\}_{a\in\Gamma}$ can be used to extend the representations of Theorem 3.1.4 from $\mathfrak{n}_+$ and $\mathfrak{n}_-$ to X- and Y-square nilpotent representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$.*

Proof. We use Corollary 3.2.4 to show $\{\mu_a\}_{a\in\Gamma}$ is a component weight function. Let $b \in \Gamma$, let $(F, I) \in \mathcal{FI}(P)$, and let x be minimal in F. Define $a := \kappa(x)$. We must show $\mu_b(F - x, I + x) - \mu_b(F, I) = \theta_{ab}$.

First suppose $a = b$. Start with the case $P_a \cap I = \emptyset$. Since x is minimal in F, we have $\psi_a(F, I) = |\Psi_a(F, I)| = 0$ and $\mu_a(F, I) = -1$. Here $P_a \cap (I + x) \neq \emptyset$. Since x is maximal in $I + x$, we have $\upsilon_a(F - x, I + x) = |\Upsilon_a(F - x, I + x)| = 0$ and $\mu_a(F - x, I + x) = 1$. We get $\mu_a(F - x, I + x) - \mu_a(F, I) = 2 = \theta_{aa}$. Otherwise we have the case $P_a \cap I \neq \emptyset$. Let $z \in P_a \cap I$ and note that $P_a \cap [z, x]$ is finite by local finiteness for $[z, x]$. So $P_a \cap I$ has a maximal element y. Here $y < x$ are consecutive occurrences of the color a. By I2A there are exactly two elements $u, v \in (y, x)$ with colors adjacent to a. By AC all elements greater than y in I with colors adjacent to a are in (y, x). Hence u and v are the only such elements. This shows both u and v are in $\Upsilon_a(F, I)$, and no other elements can be in $\Upsilon_a(F, I)$. Thus $\upsilon_a(F, I) = |\Upsilon_b(F, I)| = 2$ and $\mu_a(F, I) = -1$. We still have $\mu_a(F - x, I + x) = 1$, and so again $\mu_a(F - x, I + x) - \mu_a(F, I) = 2 = \theta_{aa}$.

Now suppose $a \sim b$. Again start with the case $P_b \cap I = \emptyset$. Here $P_b \cap (I + x) = \emptyset$ as well. We know $P_b \cap F \neq \emptyset$ since κ is surjective. Let $z \in P_b \cap F$. Note $x < z$ by AC and that $[x, z]$ and $P_b \cap [x, z]$ are finite. So $P_b \cap F$ has a minimal element y. Then x satisfies all three criteria to be in $\Psi_b(F, I)$. But $x \notin \Psi_b(F - x, I + x)$ since $x \notin F - x$. Observe that y is also minimal in $P_b \cap (F - x)$. So we have $\Psi_b(F - x, I + x) = \Psi_b(F, I) - \{x\}$. Thus $|\Psi_b(F - x, I + x)| = |\Psi_b(F, I)| - 1$. Note that there is some color $c \sim b$ such that there are infinitely many elements less than y in $P_c \cap F$ if and only if the same statement is true for $F - x$. Thus whether or not such a color c exists we have $\psi_b(F - x, I + x) = \psi_b(F, I) - 1$. Here $\mu_b(F - x, I + x) = -1 + \psi_b(F - x, I + x)$ and $\mu_b(F, I) = -1 + \psi_b(F, I)$. Thus $\mu_b(F - x, I + x) - \mu_b(F, I) = -1 = \theta_{ab}$.

Otherwise for $a \sim b$ we have the case $P_b \cap I \neq \emptyset$. Here $P_b \cap (I + x) \neq \emptyset$ as well. Let $z \in P_b \cap I$. Note $z < x$ by AC and that $[z, x]$ and $P_b \cap [z, x]$ are finite. So $P_b \cap I$ has a maximal element y. Note that x satisfies all three criteria to be in $\Upsilon_b(F - x, I + x)$. But $x \notin \Upsilon_b(F, I)$ since $x \notin I$. Observe that y is also maximal in $P_b \cap (I + x)$. So we have $\Upsilon_b(F - x, I + x) = \Upsilon_b(F, I) \cup \{x\}$. Thus $|\Upsilon_b(F - x, I + x)| = |\Upsilon_b(F, I)| + 1$. Note that there is some color $c \sim b$ such that there are infinitely many elements greater than y in $P_c \cap I$ if and only if the same statement is true for $I + x$. Thus whether or not such a color c exists we have $\upsilon_b(F - x, I + x) = \upsilon_b(F, I) + 1$. Here $\mu_b(F - x, I + x) = 1 - \upsilon_b(F - x, I + x)$ and $\mu_b(F, I) = 1 - \upsilon_b(F, I)$. Thus $\mu_b(F - x, I + x) - \mu_b(F, I) = -1 = \theta_{ab}$.

Finally suppose $a \not\sim b$. Again start with the case $P_b \cap I = \emptyset$. Here $P_b \cap (I + x) = \emptyset$ as well. An element y is minimal in $P_b \cap F$ if and only if it is minimal in $P_b \cap (F - x)$. If no such minimal element exists, then $\psi_b(F, I) = 1 = \psi_b(F - x, I + x)$. Otherwise there is an element y minimal in both $P_b \cap F$ and $P_b \cap (F - x)$. Since $a \not\sim b$, there is a color $c \sim b$ such that there are infinitely many elements less than y in $P_c \cap F$ if and only if the same statement is true for $P_c \cap (F - x)$. Thus whether or not such a color c exists we have $\psi_b(F - x, I + x) = \psi_b(F, I)$. Hence $\mu_b(F - x, I + x) - \mu_b(F, I) = 0 = \theta_{ab}$. Otherwise for $a \not\sim b$ we have the case $P_b \cap I \neq \emptyset$. Dualize the above argument with υ_b replacing ψ_b to again get $\mu_b(F - x, I + x) - \mu_b(F, I) = 0 = \theta_{ab}$.

So $\{\mu_a\}_{a\in\Gamma}$ is a component weight function. Since I2A implies I3ND, we can apply Corollary 3.2.6(b) to get the last statement. □

4.2 Existence and uniqueness for $\mathfrak{sl}_2$ weights along color strings

The actions of the $\{h_a\}_{a\in\Gamma}$ in a minuscule representation of a semisimple Lie algebra have certain values along their "$\mathfrak{sl}_2$ strings." To obtain upper P-minuscule representations of $\mathfrak{b}'_+$ and P-minuscule representations of $\mathfrak{g}'$, we need component weight functions that have these values along the "color strings" of $\mathcal{FI}(P)$. The next result is the first step toward obtaining these values. This existence result motivates the properties AC and I2A from a Lie representation viewpoint.

Proposition 4.2.1. *The following are equivalent:*

(i) There exists a component weight function $\{\eta_a\}_{a\in\Gamma}$ such that for every $b \in \Gamma$ and every split (F, I), we have $\eta_b(F, I) = -1$ if b is the color of a minimal element of F.

(ii) The properties EC, AC, and I2A are satisfied by P.

(iii) There exists a component weight function $\{\eta_a\}_{a\in\Gamma}$ such that for every $b \in \Gamma$ and every split (F, I), we have $\eta_b(F, I) = +1$ if b is the color of a maximal element of I.

If these conditions are satisfied, then any choice of component weight function made for Part (i) will work for Part (iii) (and vice versa). The component weight function $\{\mu_a\}_{a\in\Gamma}$ of Proposition 4.1.2 satisfies the conditions of Parts (i) and (iii).

For each $a \in \Gamma$, Part (i) (or (iii)) requires certain values for the function η_a along the edges of color a in $\mathcal{FI}(P)$. But when $\mathcal{FI}(P)$ has a component that does not contain an edge of some color $b \in \Gamma$, those parts do not pertain to η_b on that component. The component containing (F, I) in Figure 4.1 does not contain an edge colored g. So there $\eta_g(F, I)$ can be any complex number.

Proof. We first show (iii) implies (i). Create a component weight function $\{\eta_a\}_{a\in\Gamma}$ that satisfies (iii). Let $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$. Suppose b is the color of a minimal element y of F. Then $\eta_b(F - y, I + y) = 1$ by (iii) since y is maximal in $I + y$. Equation (3.2) gives $\eta_b(F - y, I + y) - \eta_b(F, I) = 2$. Hence we get $\eta_b(F, I) = \eta_b(F - y, I + y) - 2 = -1$, yielding (i). Dualize to get (i) implies (iii). This also shows that one choice will work for both Parts (i) and (iii).

We next show (iii) implies (ii). Continue to consider the $\{\eta_a\}_{a\in\Gamma}$ above. Let x and y be incomparable elements in P. Define $a := \kappa(x)$ and $b := \kappa(y)$. Let F be the filter generated by x and y and set $I := P - F$. Note that b is the color of a maximal element of both $I+y$ and $I+y+x$. Thus by (iii) we have $\eta_b(F-y, I+y) = 1 = \eta_b(F - y - x, I + y + x)$, so $\eta_b(F - y - x, I + y + x) - \eta_b(F - y, I + y) = 0$. Since x is minimal in $F - y$, we can apply Equation (3.2) to also obtain $\eta_b(F - y - x, I + y + x) - \eta_b(F - y, I + y) = \theta_{ab}$. Thus $\theta_{ab} = 0$, and so $a \not\sim b$. Thus we get both EC and AC. Now let $b \in \Gamma$ and let $x < y$ be consecutive occurrences of the color b. Define I' to be the principal ideal generated by y. Define an ideal I to be $I' - (x, y]$, where $(x, y] := \{z \in P \mid x < z \leq y\}$. Also note that x is maximal in I. Define $F' := P - I'$ and $F := P - I$. Since $(x, y]$ is finite, the splits (F', I') and (F, I) are in the same component of $\mathcal{FI}(P)$. Since y is maximal in I' and x is maximal in I, we have $\eta_b(F', I') = 1 = \eta_b(F, I)$. Also note $I' - I = (x, y]$ and $I - I' = \emptyset$. Thus using Equation (3.1) we get

$$\begin{aligned} 0 = \eta_b(F', I') - \eta_b(F, I) &= 2\Delta_b[(F', I'), (F, I)] - \sum_{c\sim b} \Delta_c[(F', I'), (F, I)] \\ &= 2|P_b \cap (x, y]| - \sum_{c\sim b} |P_c \cap (x, y]|. \end{aligned}$$

Since $\kappa(y) = b$ this equation can be rewritten $2 = \sum_{c\sim b} |P_c \cap (x, y)|$. Thus I2A holds.

Now assume (ii) holds and consider the weight function $\{\mu_a\}_{a\in\Gamma}$. By Proposition 4.1.2 it is a component weight function. Let $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$. Assume b is the color of a maximal element of I. Then we have $\upsilon_b(F, I) = 0$ and $\mu_b(F, I) = +1$, so (iii) holds. □

We do get uniqueness for $\{\mu_a\}_{a\in\Gamma}$ on a component when the component has edges of all colors:

Corollary 4.2.2. *Suppose P satisfies EC, AC, and I2A. Fix a component C and suppose there is an edge in C of every color. Then $\{\mu_a\}_{a\in\Gamma}$ is the unique restriction to C of a component weight function that satisfies Part (i) or Part (iii) of Proposition 4.2.1. So if P is finite, then $\{\mu_a\}_{a\in\Gamma}$ is the unique component weight function on $\mathcal{FI}(P)$ that satisfies Proposition 4.2.1.*

Proof. Create a component weight function $\{\eta_a\}_{a\in\Gamma}$ that satisfies Part (i) of Proposition 4.2.1. Let $b \in \Gamma$. Since there is an edge in C of color b, let $(F_0, I_0) \in C$ be such that F_0 has a minimal element y of color b. We know $\eta_b(F_0, I_0) = -1$. The last statement of Proposition 4.2.1 says $\mu_b(F_0, I_0) = -1$ as well. Now Lemma 3.2.7(b) says that η_b and μ_b agree on all of C. Apply similar reasoning when $\{\eta_a\}_{a\in\Gamma}$ satisfies Part (iii) of Proposition 4.2.1. When P is finite, the lattice $\mathcal{FI}(P)$ has one component and $\kappa : P \to \Gamma$ is surjective. □

4.3 Frontier census properties and eigenvalue bounds

Here we introduce our last coloring properties; they limit the eigenvalues of the actions of coroots. For each $k \geq 1$ we define two *frontier census properties*:

(MxkGA): For every color $a \in \Gamma$: If x is maximal in P_a, then there are at most k elements greater than x that have their colors adjacent to a,

(MnkLA): For every color $a \in \Gamma$: If x is minimal in P_a, then there are at most k elements less than x that have their colors adjacent to a.

We also introduce two more general such properties:

(MxFGA): For every color $a \in \Gamma$: If x is maximal in P_a, then the number of elements greater than x that have their colors adjacent to a is finite,

(MnFLA): For every color $a \in \Gamma$: If x is minimal in P_a, then the number of elements less than x that have their colors adjacent to a is finite.

The properties Mx1GA and Mn1LA are the most important of these properties. They will be used to finish the characterizations of (upper) P-minuscule representations of $\mathfrak{g}'$ (respectively $\mathfrak{b}'_+$) in Chapter 5. However, their primary importance comes from revamping, generalizing, and unifying axioms considered by Stembridge and Green. See the paragraph following the extended definitions of the frontier census properties in Section 4.6 for more on this.

The three results below describe the interactions between the frontier census properties and the component weight function $\{\mu_a\}_{a\in\Gamma}$ constructed in Section 4.1. Define $\mathcal{E}_\mu := \{\mu_a(F, I) \mid a \in \Gamma, (F, I) \in \mathcal{FI}(P)\}$.

Proposition 4.3.1. *Suppose P satisfies EC, AC, and I2A. Then for every integer $k \geq 2$:*

(a) The number $1 - k$ is a lower bound for $\mathcal{E}_\mu$ if and only if P satisfies MxkGA.

(b) The number $k - 1$ is an upper bound for $\mathcal{E}_\mu$ if and only if P satisfies MnkLA.

Proof. Fix an integer $k \geq 2$. Suppose there is some $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$ such that $\mu_b(F, I) < 1 - k$.

Since $k \geq 2$, we get $\mu_b(F, I) < -1$. We have $P_b \cap I \neq \emptyset$ since $\mu_b(F, I) \geq -1$ whenever $P_b \cap I = \emptyset$. Hence $\mu_b(F, I) = 1 - \upsilon_b(F, I)$, so $\upsilon_b(F, I) > k$. This shows $P_b \cap I$ has a maximal element y since $\upsilon_b(F, I) = 1$ when $P_b \cap I$ has no maximal element. Let $c \in \Gamma$ be such that $c \sim b$ and suppose there are infinitely many elements greater than y in $P_c \cap I$. Then y must be maximal in P_b by local finiteness and AC. Thus MxkGA fails and we are done. Otherwise we have $|\Upsilon_b(F, I)| = \upsilon_b(F, I) > k$. Each element in $\Upsilon_b(F, I)$ is greater than y and has color adjacent to b. So by AC each element in $\Upsilon_b(F, I)$ is less than any element in $P_b \cap F$. But since $|\Upsilon_b(F, I)| > 2$, by I2A we see $P_b \cap F = \emptyset$. Thus y is maximal in P_b. Thus MxkGA fails again since $\Upsilon_b(F, I)$ contains more than k elements.

Now suppose MxkGA does not hold for some $k \geq 2$. Then there is some $b \in \Gamma$ for which P_b contains a maximal element y such that there are more than k elements greater than y with colors adjacent to b. Let I be an ideal generated by $k + 1$ of these elements and let $F := P - I$. By local finiteness, the intervals between y and these $k + 1$ elements are all finite. Thus for all $c \sim b$ there are finitely many elements greater than y in $P_c \cap I$. So each of these $k + 1$ elements is in $\Upsilon_b(F, I)$ and we have $\upsilon_b(F, I) = |\Upsilon_b(F, I)| > k$. Hence $\mu_b(F, I) < 1 - k$.

Dualize to get (b). □

We use this result to get:

Corollary 4.3.2. *Suppose P satisfies EC, AC, and I2A.*

(a) We have $\mathcal{E}_\mu \subseteq \{-1, 0, 1\}$ if and only if P satisfies Mx2GA and Mn2LA.

(b) The set $\mathcal{E}_\mu$ is finite if and only if P satisfies MxFGA and MnFLA.

(c) Let $\{\eta_a\}_{a \in \Gamma}$ be any component weight function and let $\mathcal{E}_\eta := \{\eta_a(F, I) \mid a \in \Gamma, (F, I) \in \mathcal{FI}(P)\}$. Then the set $\mathcal{E}_\eta$ is finite if and only if P satisfies MxFGA and MnFLA.

Proof. Part (a) follows immediately from Proposition 4.3.1 since $\mathcal{E}_\mu \subseteq \mathbb{Z}$. For any $k \geq 1$, the property MxkGA (respectively MnkLA) implies MxFGA (respectively MnFLA). Conversely, the property MxFGA (respectively MnFLA) implies the existence of some $l \geq 1$ such that MxkGA (respectively MnkLA) holds for all $k \geq l$. Combined with both parts of Proposition 4.3.1, this gives (b). For (c), let C be a component of $\mathcal{FI}(P)$ and fix $(F_0, I_0) \in C$. Let $b \in \Gamma$ and let $(F, I) \in C$. Since both $\{\mu_a\}_{a \in \Gamma}$ and $\{\eta_a\}_{a \in \Gamma}$ are component weight functions, applying Equation (3.1) twice gives

$$\mu_b(F, I) - \mu_b(F_0, I_0) = 2\Delta_b[(F, I), (F_0, I_0)] - \sum_{c \sim b} \Delta_c[(F, I), (F_0, I_0)] = \eta_b(F, I) - \eta_b(F_0, I_0).$$

Thus $\mu_b(F, I) - \eta_b(F, I) = \mu_b(F_0, I_0) - \eta_b(F_0, I_0)$, so the difference $\mu_b - \eta_b$ is constant on C. Since this holds for all components, we see that $\mathcal{E}_\mu$ is finite if and only if $\mathcal{E}_\eta$ is finite. Thus (c) follows from (b). □

We close by specifying when the bounds for $\mathcal{E}_\mu$ are attained:

Corollary 4.3.3. *Suppose P satisfies EC, AC, and I2A.*

(a) Suppose P additionally satisfies MxFGA. Let k be the smallest positive integer for which P satisfies MxkGA. If $k = 1$, then $\min \mathcal{E}_\mu = -1$. *If $k > 1$, then* $\min \mathcal{E}_\mu = 1 - k$.

(b) Suppose P additionally satisfies MnFLA. Let k be the smallest positive integer for which P satisfies MnkLA. If $k = 1$, then $\max \mathcal{E}_\mu = 1$. *If $k > 1$, then* $\max \mathcal{E}_\mu = k - 1$.

Proof. To prove (a), first suppose $k = 1$. Since Mx1GA implies Mx2GA, it follows from Proposition 4.3.1 that -1 is a lower bound for $\mathcal{E}_\mu$. Let $y \in P$ and set $b := \kappa(y)$. Let F be the principal filter generated by y and set $I := P - F$. By the last statement of Proposition 4.2.1 we know that $\mu_b(F, I) = -1$. Hence $-1 \in \mathcal{E}_\mu$, so $\min \mathcal{E}_\mu = -1$. Now suppose $k > 1$. The same argument as above shows that $-1 \in \mathcal{E}_\mu$. By Proposition 4.3.1 we know that $1 - k$ is a lower bound for $\mathcal{E}_\mu$. By assumption P does not satisfy MxlGA for any $1 \leq l < k$. So Proposition 4.3.1 implies $1 - l$ is not a lower bound for $\mathcal{E}_\mu$ for any $1 < l < k$. Since $-1 \in \mathcal{E}_\mu$, we see $1 - l$ is not a lower bound for $\mathcal{E}_\mu$ for any $1 \leq l < k$. Since $\mathcal{E}_\mu \subseteq \mathbb{Z}$, this implies that $1 - k \in \mathcal{E}_\mu$. Thus $\min \mathcal{E}_\mu = 1 - k$. Dualize to get (b). □

4.4 A combinatorially motivated component weight function in the general case

We begin to extend the results of Chapter 4 to the general case. See Section 2.5 for a description of numbering conventions and a summary of changes between the simply laced and general cases. Here we refer to the extended definitions for the general case made in Section 2.4 and the general case definitions and results of Sections 3.3 and 3.4.

We continue to assume P satisfies EC. To extend the definition of the weight function $\{\mu_a\}_{a \in \Gamma}$ to the general case, we first update the auxiliary functions $\{\upsilon_a\}_{a \in \Gamma}$ and $\{\psi_a\}_{a \in \Gamma}$ from $\mathcal{FI}(P)$ to $\mathbb{N}$. Fix a color $b \in \Gamma$. Let (F, I) be a split. If $P_b \cap I$ does not have a maximal element, then again set $\upsilon_b(F, I) := 1$. Now suppose that $P_b \cap I$ has a maximal element y. By EC the element y is unique. We define $\Upsilon_b(F, I)$ exactly as in the simply laced case: Let $z \in I$. We place z into $\Upsilon_b(F, I)$ if it meets the following three requirements:

(i) The element z is greater than y,

(ii) Its color $c := \kappa(z)$ is adjacent to b, and

(iii) The number of elements greater than y that are in $P_c \cap I$ is finite.

Since Γ is finite, by (iii) we see $\Upsilon_b(F, I)$ is finite. Hence the sum $\sum_{z\in\Upsilon_b(F,I)} -\theta_{\kappa(z),b}$ is always finite. If there is some color $a \sim b$ such that there are infinitely many elements greater than y in $P_a \cap I$, then set $\upsilon_b(F, I) := 1 + \sum_{z\in\Upsilon_b(F,I)} -\theta_{\kappa(z),b}$. Otherwise set $\upsilon_b(F, I) := \sum_{z\in\Upsilon_b(F,I)} -\theta_{\kappa(z),b}$. Each summand is 1 when Γ is simply laced, so this case reduces to the original definition of $\upsilon_b(F, I)$.

Next we update $\psi_b : \mathcal{FI}(P) \to \mathbb{N}$. Let (F, I) be a split. If $P_b \cap F$ does not have a minimal element, then again set $\psi_b(F, I) := 1$. Now suppose that $P_b \cap F$ has a minimal element y. By EC the element y is unique. We also define $\Psi_b(F, I)$ exactly as in the simply laced case. The sum $\sum_{z\in\Psi_b(F,I)} -\theta_{\kappa(z),b}$ is also finite. If there is some color $a \sim b$ such that there are infinitely many elements less than y in $P_a \cap F$, then set $\psi_b(F, I) := 1 + \sum_{z\in\Psi_b(F,I)} -\theta_{\kappa(z),b}$. Otherwise set $\psi_b(F, I) := \sum_{z\in\Psi_b(F,I)} -\theta_{\kappa(z),b}$. This again reduces to the original definition of $\psi_b(F, I)$ when Γ is simply laced.

Now define $\mu_b : \mathcal{FI}(P) \to \mathbb{Z}$ as before: Let $(F, I) \in \mathcal{FI}(P)$. If $P_b \cap I \neq \emptyset$, then set $\mu_b(F, I) := 1 - \upsilon_b(F, I)$. If $P_b \cap I = \emptyset$, then set $\mu_b(F, I) := -1 + \psi_b(F, I)$. Define the Γ-set of operators $\{M_a\}_{a\in\Gamma}$ to be the diagonal operators with $\mathfrak{h}'$-weight $\{\mu_a\}_{a\in\Gamma}$. These are still called the *μ-diagonal operators* and reduce to the original definition when Γ is simply laced. Proposition 4.1.1 is not used, so we omit its extension to the general case. We extend the property I2A in order to update Proposition 4.1.2 in the general case:

Proposition 4.4.2. *Suppose P satisfies EC, AC, and the following additional property:*

(I2$\vee$1A): For every $a \in \Gamma$: The open interval between any two consecutive elements of color a either: (i) contains exactly two elements whose colors are adjacent to a, and these colors are 1*-adjacent to a, or (ii) contains exactly one element whose color is adjacent to a, and this color is* 2*-adjacent to a.*

Then the weight function $\{\mu_a\}_{a\in\Gamma}$ is a component weight function. If P further satisfies NA, then the corresponding operators $\{M_a\}_{a\in\Gamma}$ can be used to extend the representations of Theorem 3.3.4 from $\mathfrak{n}_+$ and $\mathfrak{n}_-$ to X- and Y-square nilpotent representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$.

The property I2$\vee$1A is very similar to Stembridge's property H2 [Ste]. However, Part (ii) of I2$\vee$1A allows for more than one element in this interval, while H2 requires the interval to contain only one element. In the forthcoming Proposition 7.4.2, we show that these properties are equivalent when neighbors are required to have weakly adjacent colors. That proposition also shows I2∨1A is equivalent to Green's Property F3 [Gr3] without the additional requirement. For reference, there we have renamed Stembridge's property S2 and Green's property G5. This version of the property I2∨1A implies both ND and I3NE.

Our proof of Proposition 4.4.2 follows the same structure as the original proof.

Proof. We use Corollary 3.4.4 to show $\{\mu_a\}_{a\in\Gamma}$ is a component weight function. Let $b \in \Gamma$, let $(F, I) \in \mathcal{FI}(P)$, and let x be minimal in F. Define $a := \kappa(x)$. We must show $\mu_b(F - x, I + x) - \mu_b(F, I) = \theta_{ab}$.

First suppose $a = b$. Start with the case $P_a \cap I = \emptyset$. Note that $\Psi_a(F, I) = \emptyset$ since x is minimal in F, so $\psi_a(F, I) = \sum_{z\in\Psi_a(F,I)} -\theta_{\kappa(z),a} = 0$. We also have $\Upsilon_a(F - x, I + x) = \emptyset$ since x is maximal in $I + x$, so $\upsilon_a(F - x, I + x) = 0$. Hence $\mu_a(F - x, I + x) = 1$ and $\mu_a(F, I) = -1$, so $\mu_a(F - x, I + x) - \mu_a(F, I) = 2 = \theta_{aa}$. Otherwise we have the case $P_a \cap I \neq \emptyset$. Then $P_a \cap I$ has a maximal element y by local finiteness since $P_a \cap F \neq \emptyset$. Here $y < x$ are consecutive occurrences of the color a. Consider the two finite sums $\sum_{z\in\Upsilon_a(F,I)} -\theta_{\kappa(z),a}$ and $\sum_{z\in(y,x)} -\theta_{\kappa(z),a}$. The summands in each consist of non-negative integers. By AC we see that $\Upsilon_a(F, I) \subseteq (y, x)$, and so $\sum_{z\in\Upsilon_a(F,I)} -\theta_{\kappa(z),a} \leq \sum_{z\in(y,x)} -\theta_{\kappa(z),a}$. Now suppose $u \in (y, x)$ and define $c := \kappa(u)$. If $c \nsim a$, then $-\theta_{\kappa(u),a} = 0$. Now suppose $c \sim a$. Since x is minimal in F, we see that $u \in I$. All elements of color c greater than y in I must also be in (y, x) by AC. Since this interval is finite, we see that $u \in \Upsilon_a(F, I)$. Thus $\sum_{z\in(y,x)} -\theta_{\kappa(z),a} \leq \sum_{z\in\Upsilon_a(F,I)} -\theta_{\kappa(z),a}$, and so these two sums are equal. By I2∨1A, we then get $2 = \sum_{z\in(y,x)} -\theta_{\kappa(z),a} = \sum_{z\in\Upsilon_a(F,I)} -\theta_{\kappa(z),a}$. There is no color $d \sim a$ such that there are infinitely many elements greater than y in $P_d \cap I$, since each such element must be in the finite interval (y, x) by AC. Hence we have $\upsilon_a(F, I) = \sum_{z\in\Upsilon_a(F,I)} -\theta_{\kappa(z),a} = 2$ and $\mu_a(F, I) = -1$. We still have $\mu_a(F - x, I + x) = 1$, and so again $\mu_a(F - x, I + x) - \mu_a(F, I) = 2 = \theta_{aa}$.

Now suppose a is k-adjacent to b for some $k \geq 1$. Then $\theta_{\kappa(x),b} = \theta_{ab} = -k$. Start with the case $P_b \cap I = \emptyset$. Here $P_b \cap (I + x) = \emptyset$ as well. We know $P_b \cap F \neq \emptyset$ since κ is surjective. Let $z \in P_b \cap F$. Note $x < z$ by AC and that $[x, z]$ and $P_b \cap [x, z]$ are finite. So $P_b \cap F$ has a minimal element y. Then x satisfies all three criteria to be in $\Psi_b(F, I)$. But $x \notin \Psi_b(F - x, I + x)$ since $x \notin F - x$. Observe that y is also minimal in $P_b \cap (F - x)$. So we have $\Psi_b(F - x, I + x) = \Psi_b(F, I) - \{x\}$. This implies $\sum_{z\in\Psi_b(F-x,I+x)} -\theta_{\kappa(z),b} = \left(\sum_{z\in\Psi_b(F,I)} -\theta_{\kappa(z),b}\right) + \theta_{\kappa(x),b}$. Note that there is some color $c \sim b$ such that there are infinitely many elements less than y in $P_c \cap F$ if and only if the same statement is true for $F - x$. Thus whether or not such a color c exists we have $\psi_b(F - x, I + x) = \psi_b(F, I) + \theta_{\kappa(x),b} = \psi_b(F, I) - k$. Here $\mu_b(F - x, I + x) = -1 + \psi_b(F - x, I + x)$ and

$\mu_b(F, I) = -1 + \psi_b(F, I)$. Thus $\mu_b(F - x, I + x) - \mu_b(F, I) = -k = \theta_{ab}$.

Otherwise when a is k-adjacent to b we have $P_b \cap I \neq \emptyset$. Here $P_b \cap (I + x) \neq \emptyset$ as well. Let $z \in P_b \cap I$. Note $z < x$ by AC and that $[z, x]$ and $P_b \cap [z, x]$ are finite. So $P_b \cap I$ has a maximal element y. Note that x satisfies all three criteria to be in $\Upsilon_b(F - x, I + x)$. But $x \notin \Upsilon_b(F, I)$ since $x \notin I$. Observe that y is also maximal in $P_b \cap (I + x)$. So we have $\Upsilon_b(F - x, I + x) = \Upsilon_b(F, I) \cup \{x\}$. This implies $\sum_{z \in \Upsilon_b(F-x,I+x)} -\theta_{\kappa(z),b} = \left(\sum_{z \in \Upsilon_b(F,I)} -\theta_{\kappa(z),b}\right) - \theta_{\kappa(x),b}$. Note that there is some color $c \sim b$ such that there are infinitely many elements greater than y in $P_c \cap I$ if and only if the same statement is true for $I + x$. Thus whether or not such a color c exists we have $\upsilon_b(F - x, I + x) = \upsilon_b(F, I) - \theta_{\kappa(x),b} = \upsilon_b(F, I) + k$. Here $\mu_b(F - x, I + x) = 1 - \upsilon_b(F - x, I + x)$ and $\mu_b(F, I) = 1 - \upsilon_b(F, I)$. Thus $\mu_b(F - x, I + x) - \mu_b(F, I) = -k = \theta_{ab}$.

Finally suppose $a \nsim b$. The original proof of this case can be applied verbatim.

So $\{\mu_a\}_{a \in \Gamma}$ is a component weight function. Since I2∨1A implies I3NE, we can apply Corollary 3.4.6(b) to get the last statement. □

4.5 Existence and uniqueness for $\mathfrak{sl}_2$ weights along color strings in the general case

As we extend the results of Section 4.2 to the general case, we no longer assume P satisfies EC. In the statement of Proposition 4.2.1, we only need to update I2A to I2∨1A:

Proposition 4.5.1. *The following are equivalent:*

(i) There exists a component weight function $\{\eta_a\}_{a \in \Gamma}$ such that for every $b \in \Gamma$ and every split (F, I), we have $\eta_b(F, I) = -1$ if b is the color of a minimal element of F.

(ii) The properties EC, AC, and I2∨1A are satisfied by P.

(iii) There exists a component weight function $\{\eta_a\}_{a \in \Gamma}$ such that for every $b \in \Gamma$ and every split (F, I), we have $\eta_b(F, I) = +1$ if b is the color of a maximal element of I.

If these conditions are satisfied, then any choice of component weight function made for Part (i) will work for Part (iii) (and vice versa). The component weight function $\{\mu_a\}_{a \in \Gamma}$ of Proposition 4.4.2 satisfies the conditions of Parts (i) and (iii).

Proof. The proof that (i) is equivalent to (iii) given for Proposition 4.2.1 in Section 4.2 applies verbatim here. That argument still shows that one choice will work for both Parts (i) and (iii).

We next show (iii) implies (ii). Create a component weight function $\{\eta_a\}_{a \in \Gamma}$ that satisfies (iii). The argument to derive EC and AC in the proof of Proposition 4.2.1 in Section 4.2 applies verbatim here. Now let

$b \in \Gamma$ and let $x < y$ be consecutive occurrences of the color b. Define splits (F, I) and (F', I') as before. As before, we have $\eta_b(F', I') = 1 = \eta_b(F, I)$, with $I' - I = (x, y]$ and $I - I' = \emptyset$. Then using Equation (3.6) we get

$$0 = \eta_b(F', I') - \eta_b(F, I) = \sum_{c \in \Gamma} \theta_{cb} \Delta_c[(F', I'), (F, I)] = \sum_{c \in \Gamma} \theta_{cb} |P_c \cap (x, y]|.$$

Since $\kappa(y) = b$ and $\theta_{bb} = 2$, this equation can be rewritten $2 = \sum_{c \in \Gamma - \{b\}} -\theta_{cb} |P_c \cap (x, y)|$. Note that $-\theta_{cb} \geq 0$ for all $c \neq b$, so the sum on the right-hand side consists of non-negative integers. Suppose the right-hand side is $1 + 1$. Then there are distinct colors e and f that are 1-adjacent to b with $|P_e \cap (x, y)| = 1 = |P_f \cap (x, y)|$. Also, we have $P_c \cap (x, y) = \emptyset$ for all other colors $c \sim b$, so we are in the situation of I2$\vee$1A(i). Now suppose the right-hand side is $1 \cdot 2$. Then there is one color a that is 1-adjacent to b with $|P_a \cap (x, y)| = 2$. Also, we have $P_c \cap (x, y) = \emptyset$ for all other colors $c \sim b$, so we are again in the situation of I2$\vee$1A(i). Now suppose the right-hand side is $2 \cdot 1$. Then there is one color d that is 2-adjacent to b with $|P_b \cap (x, y)| = 1$. Also, we have $P_c \cap (x, y) = \emptyset$ for all other colors $c \sim b$, so we are in the situation of I2$\vee$1A(ii). So in every case I2$\vee$1A is satisfied.

The proof of (ii) implies (iii) for Proposition 4.2.1 in Section 4.2 applies verbatim. $\square$

In the statement of Corollary 4.2.2, we again only need to update I2A to I2$\vee$1A; here the original proof can be applied verbatim.

Corollary 4.5.2. *Suppose P satisfies EC, AC, and I2$\vee$1A. Fix a component C and suppose there is an edge in C of every color. Then $\{\mu_a\}_{a \in \Gamma}$ is the unique restriction to C of a component weight function that satisfies Part (i) or Part (iii) of Proposition 4.5.1. So if P is finite, then $\{\mu_a\}_{a \in \Gamma}$ is the unique component weight function on $\mathcal{FI}(P)$ that satisfies Proposition 4.5.1.*

4.6 Frontier census properties and eigenvalue bounds in the general case

Now we extend the frontier census bounds of Section 4.3 to the general case. For each $k \geq 1$ we extend two of the *frontier census properties*:

(MxkSB): For every color $a \in \Gamma$: If x is maximal in P_a, then the number of elements greater than x that have their colors adjacent to a is finite and $\sum_{y > x} -\theta_{\kappa(y), a} \leq k$.

(MnkSB): For every color $a \in \Gamma$: If x is minimal in P_a, then the number of elements less than x that have their colors adjacent to a is finite and $\sum_{y < x} -\theta_{\kappa(y), a} \leq k$.

The statements of MxkSB and MnkSB imply that their respective sums $\sum_{y > x} -\theta_{\kappa(y), a}$ and $\sum_{y < x} -\theta_{\kappa(y), a}$ contain only finitely many nonzero terms. We have just interpreted "$\sum$" as follows: If there are infinitely many elements y greater (or less) than x for which $\theta_{\kappa(y),a} = 0$, ignore those terms. We call the sum $\sum_{y>x} -\theta_{\kappa(y),a}$ (respectively $\sum_{y<x} -\theta_{\kappa(y),a}$) an *upper* (respectively a *lower*) *adjacency sum* since $-\theta_{\kappa(y),a} \neq 0$ if and only if $\kappa(y) \sim a$. We note that these properties have the following equivalent forms:

(MxkSB): The property MxFGA holds and for every $a \in \Gamma$: If x is maximal in P_a, then $\sum_{y>x} -\theta_{\kappa(y),a} \leq k$.

(MnkSB): The property MnFLA holds and for every $a \in \Gamma$: If x is minimal in P_a, then $\sum_{y<x} -\theta_{\kappa(y),a} \leq k$.

The properties Mx1SB and Mn1SB continue to be the most important of these properties. The property Mn2SB also plays an important role in the classification in Chapter 8; see Proposition 8.2.2. In Proposition 7.4.5 we indicate how Mx1SB revamps axioms considered by Stembridge [Ste]. This property was retrospectively found to be implicitly present in Proposition 2.5 of [Ste]. That early statement in [Ste] was formulated in terms of decompositions of Weyl group elements w, before the heap finite colored posets were introduced. Both properties Mx1SB and Mn1SB are satisfied vacuously by the full heaps of Green. So these updated frontier census bounds revamp, generalize, and unify the axioms considered by Stembridge and Green.

Continue to denote $\mathcal{E}_\mu = \{\mu_a(F, I) \mid a \in \Gamma, (F, I) \in \mathcal{FI}(P)\}$. All of the results from Section 4.3 can be extended to the general case. To update Proposition 4.3.1, just replace I2A, MxkGA, and MnkLA with I2$\vee$1A, MxkSB, and MnkSB, respectively:

Proposition 4.6.1. *Suppose P satisfies EC, AC, and I2$\vee$1A. Then for every integer $k \geq 2$:*

(a) The number $1 - k$ is a lower bound for $\mathcal{E}_\mu$ if and only if P satisfies MxkSB.

(b) The number $k - 1$ is an upper bound for $\mathcal{E}_\mu$ if and only if P satisfies MnkSB.

We provide a new proof since the computations have changed with the updated properties.

Proof. Fix an integer $k \geq 2$. Suppose there is some $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$ such that $\mu_b(F, I) < 1 - k$. Since $k \geq 2$, we get $\mu_b(F, I) < -1$. We have $P_b \cap I \neq \emptyset$ since $\mu_b(F, I) \geq -1$ whenever $P_b \cap I = \emptyset$. Hence $\mu_b(F, I) = 1 - \upsilon_b(F, I)$, so $\upsilon_b(F, I) > k$. This shows $P_b \cap I$ has a maximal element y since $\upsilon_b(F, I) = 1$ when $P_b \cap I$ has no maximal element. Let $c \in \Gamma$ be such that $c \sim b$ and suppose there are infinitely many elements greater than y in $P_c \cap I$. Then y must be maximal in P_b by local finiteness and AC. Thus MxkSB fails and we are done. Otherwise we have $\sum_{z\in\Upsilon_b(F,I)} -\theta_{\kappa(z),b} = \upsilon_b(F, I) > k \geq 2$. Each element in $\Upsilon_b(F, I) \subseteq I$ is greater than y and has color adjacent to b. Suppose $P_b \cap F \neq \emptyset$. Since y is maximal in $P_b \cap I$, we see $P_b \cap F$ must have a minimal element x by local finiteness. Note $y < x$ are consecutive occurrences of the color b. By AC each element in $\Upsilon_b(F, I)$ is in (y, x). But this violates I2$\vee$1A since it implies $\sum_{z\in(y,x)} -\theta_{\kappa(z),b} \geq \sum_{z\in\Upsilon_b(F,I)} -\theta_{\kappa(z),b} > 2$.

Thus $P_b \cap F = \emptyset$ and so y is maximal in P_b. But even if MxFGA holds, we see MxkSB still fails since then $\sum_{z>y} -\theta_{\kappa(z),b} \geq \sum_{z\in\Upsilon_b(F,I)} -\theta_{\kappa(z),b} > k$.

Now suppose MxkSB does not hold for some $k \geq 2$. Then there is a color $b \in \Gamma$ and an element y maximal in P_b such that either there are infinitely many elements greater than y with colors adjacent to b, or there are finitely many such elements and $\sum_{z>y} -\theta_{\kappa(z),b} > k$. Either way, choose a finite set $\mathcal{S}$ of elements greater than y with colors adjacent to b such that $\sum_{z\in\mathcal{S}} -\theta_{\kappa(z),b} > k$. Let I be the ideal generated by $\mathcal{S}$ and let $F := P - I$. By local finiteness, the intervals between y and the elements of $\mathcal{S}$ are all finite. Thus for all $c \sim b$ there are finitely many elements greater than y in $P_c \cap I$. Hence $\mathcal{S} \subseteq \Upsilon_b(F, I)$ and so we have $\upsilon_b(F, I) = \sum_{z\in\Upsilon_b(F,I)} -\theta_{\kappa(z),b} \geq \sum_{z\in\mathcal{S}} -\theta_{\kappa(z),b} > k$. Hence $\mu_b(F, I) < 1 - k$.

Dualize to get (b). □

To update Corollary 4.3.2, replace I2A, Mx2GA, and Mn2LA with I2∨1A, Mx2SB, and Mn2SB, respectively:

Corollary 4.6.2. *Suppose P satisfies EC, AC, and I2∨1A.*

(a) We have $\mathcal{E}_\mu \subseteq \{-1, 0, 1\}$ if and only if P satisfies Mx2SB and Mn2SB.

(b) The set $\mathcal{E}_\mu$ is finite if and only if P satisfies MxFGA and MnFLA.

(c) Let $\{\eta_a\}_{a\in\Gamma}$ be any component weight function and let $\mathcal{E}_\eta := \{\eta_a(F, I) \mid a \in \Gamma, (F, I) \in \mathcal{FI}(P)\}$. Then the set $\mathcal{E}_\eta$ is finite if and only if P satisfies MxFGA and MnFLA.

The proof of Proposition 4.3.2 from Section 4.3 applies once Equation (3.6) is referenced.

To extend Corollary 4.3.3, replace I2A, MxkGA, and MnkLA respectively with I2∨1A, MxkSB, and MnkSB:

Corollary 4.6.3. *Suppose P satisfies EC, AC, and I2∨1A.*

(a) Suppose P additionally satisfies MxFGA. Let k be the smallest positive integer for which P satisfies MxkSB. If $k = 1$, then $\min \mathcal{E}_\mu = -1$. If $k > 1$, then $\min \mathcal{E}_\mu = 1 - k$.

(b) Suppose P additionally satisfies MnFLA. Let k be the smallest positive integer for which P satisfies MnkSB. If $k = 1$, then $\max \mathcal{E}_\mu = 1$. If $k > 1$, then $\max \mathcal{E}_\mu = k - 1$.

CHAPTER 5

Minuscule representations built from posets

Here we obtain the main results making up the characterizations of the first part of this dissertation. Theorem 5.1.1 gives the characterization of upper P-minuscule representations of $\mathfrak{b}'_+$ in terms of poset coloring properties. Theorem 5.2.2 gives the characterization of P-minuscule representations of $\mathfrak{g}'$ in terms of poset coloring properties.

5.1 Upper P-minuscule representations of $\mathfrak{b}'_+$

Our first main result gives necessary and sufficient conditions on coloring properties for P so that $\mathcal{FI}(P)$ carries an upper P-minuscule representation of $\mathfrak{b}'_+$; this notion was defined at the end of Section 2.3.

Theorem 5.1.1. *Let P be a poset whose elements are colored by the nodes of a finite simple graph Γ. Let $\mathcal{FI}(P)$ be the lattice of filter-ideal splits of P. The following are equivalent:*

(i) The lattice $\mathcal{FI}(P)$ carries an upper (respectively lower) P-minuscule representation of $\mathfrak{b}'_+$ (respectively $\mathfrak{b}'_-$).

(ii) The poset P satisfies EC, NA, AC, I2A, and Mx1GA (respectively Mn1LA).

When either of these conditions is satisfied, the μ-diagonal operators $\{M_a\}_{a\in\Gamma}$ of Section 4.1 can be used to give the actions of the $\{h_a\}_{a\in\Gamma}$.

It can be confirmed that the poset displayed in Figure 2.2 satisfies the Mx1GA version of Condition (ii); the poset displayed in Figure 2.1 satisfies both versions. Once the classification of posets satisfying this collection of properties has been obtained in Theorem 8.3.8, we will show for connected P in Corollary 8.3.9 that the μ-diagonal operators are the unique operators satisfying Part (i) here (after any trivial components in $\mathcal{FI}(P)$ have been discarded).

Proof. Assume that (i) holds for $\mathfrak{b}'_+$. Since this representation is X-square nilpotent, by Theorem 3.2.8 we know P satisfies EC and NA. Let $\{H_a\}_{a\in\Gamma}$ be the diagonal operators for this representation with $\mathfrak{h}'$-weight $\{\eta_a\}_{a\in\Gamma}$ and eigenvalue set $\mathcal{E}_\eta := \{\eta_a(F,I) \mid a \in \Gamma, (F,I) \in \mathcal{FI}(P)\}$. By Corollary 3.2.6(a) we know that

$\{\eta_a\}_{a\in\Gamma}$ is a component weight function. Since the representation is upper P-minuscule, for all $a \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$:

$$\eta_a(F, I) \in \mathcal{E}_\eta \subseteq \{-1, 0, 1, 2, \dots\} \tag{5.1}$$

$$\eta_a(F, I) = -1 \text{ if and only if } a \text{ is the color of a minimal element of } F. \tag{5.2}$$

By (5.2) and Proposition 4.2.1 we see P satisfies AC and I2A. For the sake of contradiction, suppose Mx1GA fails. Then there is a color $b \in \Gamma$ and an element y maximal in P_b such that there are two or more elements greater than y with colors adjacent to b. Let u and v be two such elements. Define I' to be the ideal generated by u and v. Define $I := I' - ((y, u] \cup (y, v])$. Note I is an ideal of P and that y is maximal in I. Define $F' := P - I'$ and $F := P - I$. Further note that $I' - I = (y, u] \cup (y, v]$ and $I - I' = \emptyset$. By local finiteness, the splits (F', I') and (F, I) are in the same component of $\mathcal{FI}(P)$. Using Equation (3.1) we get

$$\begin{aligned}\eta_b(F', I') - \eta_b(F, I) &= 2\Delta_b[(F', I'), (F, I)] - \sum_{c\sim b} \Delta_c[(F', I'), (F, I)] \\ &= 2|P_b \cap ((y, u] \cup (y, v])| - \sum_{c\sim b} |P_c \cap ((y, u] \cup (y, v])|.\end{aligned}$$

Since y is maximal in P_b we have $|P_b \cap ((y, u] \cup (y, v])| = 0$. Since u and v have colors adjacent to b, we have $\sum_{c\sim b} |P_c \cap ((y, u] \cup (y, v])| \geq 2$. Thus we get the inequality $\eta_b(F', I') - \eta_b(F, I) \leq -2$. Since y is maximal in I and $\kappa(y) = b$, by the penultimate statement of Proposition 4.2.1 we get $\eta_b(F, I) = 1$. Thus the inequality becomes $\eta_b(F', I') \leq -1$. By (5.1) we have $\eta_b(F', I') \geq -1$, and so $\eta_b(F', I') = -1$. So by (5.2) we know that F' has a minimal element z of color b. Since $y \in I'$, by EC we have $y < z$. But y is maximal in P_b, so this is a contradiction.

Now assume that (ii) holds. Since P satisfies EC, AC, and I2A, we know by Proposition 4.1.2 that $\{\mu_a\}_{a\in\Gamma}$ is a component weight function. We use the operators $\{M_a\}_{a\in\Gamma}$ specified by $\{\mu_a\}_{a\in\Gamma}$ to get the desired actions of the $\{h_a\}_{a\in\Gamma}$. Using EC, NA, and I3ND (which is implied by I2A), Corollary 3.2.6(b) says $\mathcal{FI}(P)$ carries an X-square nilpotent representation of $\mathfrak{b}'_+$ when using the $\{M_a\}_{a\in\Gamma}$. The eigenvalue set $\mathcal{E}_\mu$ of these operators is contained in $\mathbb{Z}$. Let $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$. By Corollary 4.3.3(a) we know $\mu_b(F, I) \geq -1$, and so $\mathcal{E}_\mu \subseteq \{-1, 0, 1, 2, \dots\}$. The last statement of Proposition 4.2.1 says $\mu_b(F, I) = -1$ if b is the color of a minimal element of F.

To finish, we need to confirm the "only if" direction of Part (ii) of the definition of upper P-minuscule.

So suppose $\mu_b(F, I) = -1$. Start with the case $P_b \cap I \neq \emptyset$. Then $-1 = \mu_b(F, I) = 1 - \upsilon_b(F, I)$, so $\upsilon_b(F, I) = 2$. This shows $P_b \cap I$ has a maximal element y since $\upsilon_b(F, I) = 1$ otherwise. Let $c \in \Gamma$ be such that $c \sim b$. If there are infinitely many elements greater than y in $P_c \cap I$, then $P_b \cap F$ is empty by AC and local finiteness. But then y would be maximal in P_b, which would violate Mx1GA. Thus $|\Upsilon_b(F, I)| = \upsilon_b(F, I) = 2$. So there are two distinct elements u and v in I greater than y with colors adjacent to b. By Mx1GA we know that y cannot be maximal in P_b. Let $z \in P_b$ be such that $y < z$ are consecutive occurrences of the color b. Since y is maximal in $P_b \cap I$, we have $z \in F$. By AC we know $u, v \in (y, z)$. Suppose $w \in P$ with $w \to z$. Then NA implies that $\kappa(w) \sim b$. Thus by AC we have $w \in (y, z)$. By I2A we know that $w = u$ or $w = v$, so $w \in I$. Thus z is minimal in F and has color b.

Otherwise we have the case $P_b \cap I = \emptyset$. Then $-1 = \mu_b(F, I) = -1 + \psi_b(F, I)$, so $\psi_b(F, I) = 0$. This shows $P_b \cap F$ has a minimal element y since $\psi_b(F, I) = 1$ otherwise. Suppose $w \in P$ with $w \to y$. Again by NA we know that $\kappa(w) \sim b$. If there is some color $c \sim b$ such that there are infinitely many elements less than y in $P_c \cap F$, then $\psi_b(F, I) \geq 1$. Since $\psi_b(F, I) = 0$, there are only finitely many elements less than y in $P_{\kappa(w)} \cap F$. Any such elements would be in $\Psi_b(F, I)$. But since $|\Psi_b(F, I)| = \psi_b(F, I) = 0$, there cannot be such elements. Thus $w \notin \Psi_b(F, I)$, and so $w \in I$. Hence y is minimal in F and has color b. Thus $\mu_b(F, I) = -1$ if and only if b is the color of a minimal element of F. Hence this representation of $\mathfrak{b}'_+$ carried by $\mathcal{FI}(P)$ is upper P-minuscule.

Dualize to obtain the analogous equivalence for $\mathfrak{b}'_-$. $\square$

The following lemma is used to prove Corollary 5.1.3 and Theorem 5.2.2:

Lemma 5.1.2. *Let $\{H_a\}_{a \in \Gamma}$ be diagonal operators on V. Suppose the operators $\{X_a, H_a\}_{a \in \Gamma}$ satisfy HX. If the eigenvalue set of $\{H_a\}_{a \in \Gamma}$ is contained in $\{-1, 0, 1\}$, then the actions of the color raising operators $\{X_a\}_{a \in \Gamma}$ are X-square nilpotent.*

Proof. Suppose there is some $a \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$ such that $X_a^2.\langle F, I\rangle \neq 0$. Using the relation $H_a X_a = X_a H_a + [H_a, X_a] = X_a H_a + 2X_a$ obtained from HX twice, we get

$$H_a X_a^2.\langle F, I\rangle = (X_a H_a X_a + 2X_a^2).\langle F, I\rangle = (X_a^2 H_a + 2X_a^2 + 2X_a^2).\langle F, I\rangle.$$

Since the eigenvalue of $H_a.\langle F, I\rangle$ is either -1, 0, or 1, the right hand side is $\xi X_a^2.\langle F, I\rangle$ for some $\xi \in \{3, 4, 5\}$. But since all basis vectors in the expansion of $X_a^2.\langle F, I\rangle$ are $\mathfrak{h}'$-weight vectors with eigenvalues in $\{-1, 0, 1\}$, this is impossible. Thus this representation is X-square nilpotent. □

We can now obtain the equivalence of three of the conditions in Theorem 5.2.2 below:

Corollary 5.1.3. *The Conditions (ii), (iv), and (v) of Theorem 5.2.2 are equivalent. When any of these conditions are satisfied, the μ-diagonal operators $\{M_a\}_{a\in\Gamma}$ are the unique operators satisfying (ii) or (iv).*

Proof. Suppose (iv) holds with diagonal operators $\{H_a\}_{a\in\Gamma}$. The eigenvalue set of each representation is contained in $\{-1, 0, 1\}$. By Lemma 5.1.2 and its dual these representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ are respectively X- and Y-square nilpotent. The eigenvalue condition in (iv) shows for all $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$ that $H_b.\langle F, I\rangle = -\langle F, I\rangle$ (respectively $H_b.\langle F, I\rangle = +\langle F, I\rangle$) if and only if b is the color of a minimal (maximal) element of F (respectively I). Thus the representation of $\mathfrak{b}'_+$ (respectively $\mathfrak{b}'_-$) is upper (lower) P-minuscule, and so (ii) holds.

Now suppose (ii) holds with diagonal operators $\{H_a\}_{a\in\Gamma}$. Theorem 5.1.1 shows (v) holds. Since the representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ are respectively upper and lower P-minuscule, the eigenvalue set of $\{H_a\}_{a\in\Gamma}$ is contained in $\{-1, 0, 1, 2, \ldots\} \cap \{\ldots, -2, -1, 0, 1\} = \{-1, 0, 1\}$. Also, for $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$ we have $H_b.\langle F, I\rangle = -\langle F, I\rangle$ (respectively $H_b.\langle F, I\rangle = +\langle F, I\rangle$) if and only if F (respectively I) has a minimal (maximal) element of color b. Hence $H_b.\langle F, I\rangle = 0$ otherwise, so (iv) holds.

Now suppose (v) holds. Then Theorem 5.1.1 shows (ii) holds using the μ-diagonal operators $\{M_a\}_{a\in\Gamma}$.

The conditions in (iv) completely specify the operators, so they are unique. The μ-diagonal operators satisfy (ii) when these conditions hold, and this proof showed that any diagonal operators satisfying (ii) also satisfy (iv). □

5.2 P-minuscule representations of $\mathfrak{g}'$

We obtain our foremost main result, Theorem 5.2.2. It provides necessary as well as sufficient conditions on coloring properties for P needed for $\mathcal{FI}(P)$ to carry a P-minuscule representation of $\mathfrak{g}'$. Both its statement and its proof simultaneously handle finite and infinite posets.

Lemma 5.2.1. *Suppose P satisfies EC and ND. The relation $[X_b, Y_a] = 0$ holds when $a, b \in \Gamma$ are distinct.*

Proof. Fix distinct $a, b \in \Gamma$ and let $(F, I) \in \mathcal{FI}(P)$. Suppose that $X_bY_a.\langle F, I\rangle \neq 0$. Then there are distinct elements $x, y \in P$ such that y is maximal in I and x is minimal in $F + y$ with $\kappa(y) = a$ and $\kappa(x) = b$ such that $X_bY_a.\langle F, I\rangle = \langle (F + y) - x, (I - y) + x\rangle$. Since $x \neq y$ and x is minimal in $F + y$, we see x is also minimal in F. Since x and y are minimal elements of the same filter, they are incomparable. Thus we have $Y_a X_b.\langle F, I\rangle = \langle (F - x) + y, (I + x) - y\rangle = X_b Y_a.\langle F, I\rangle$. Hence $[X_b, Y_a].\langle F, I\rangle = 0$. Dualize to handle the case $Y_a X_b.\langle F, I\rangle \neq 0$. $\square$

Here we characterize the P-minuscule representations of $\mathfrak{g}'$ in several ways; see Section 2.2 for the definitions of X- and Y-square nilpotent actions and Section 2.3 for the definitions of (upper and lower) P-minuscule representations of $\mathfrak{g}'$ (and $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$).

Theorem 5.2.2. *Let P be a poset whose elements are colored by the nodes of a finite simple graph Γ. Let $\mathcal{FI}(P)$ be the lattice of filter-ideal splits of P. The following are equivalent:*

(i) The lattice $\mathcal{FI}(P)$ carries a P-minuscule representation of $\mathfrak{g}'$.

(ii) The lattice $\mathcal{FI}(P)$ carries upper and lower P-minuscule representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ respectively, using the same diagonal operators $\{H_a\}_{a\in\Gamma}$.

(iii) The lattice $\mathcal{FI}(P)$ carries X- and Y-square nilpotent representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ respectively, using diagonal operators $\{H_a\}_{a\in\Gamma}$ that satisfy $[X_a, Y_a] = H_a$ for every color $a \in \Gamma$.

(iv) The lattice $\mathcal{FI}(P)$ carries representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ with diagonal operators $\{H_a\}_{a\in\Gamma}$ that satisfy for every $b \in \Gamma$ and every split (F, I):

$H_b.\langle F, I\rangle = -\langle F, I\rangle$ *if b is the color of a minimal element of F,*

$H_b.\langle F, I\rangle = +\langle F, I\rangle$ *if b is the color of a maximal element of I, and*

$H_b.\langle F, I\rangle = 0$ *otherwise.*

(v) The poset P satisfies the properties EC, NA, AC, I2A, Mx1GA, and Mn1LA.

When any of these conditions are satisfied, the μ-diagonal operators $\{M_a\}_{a\in\Gamma}$ of Section 4.1 are the unique diagonal operators satisfying (i), (ii), (iii), or (iv).

It can be confirmed that the poset displayed in Figure 2.1 satisfies Condition (v).

Proof. Suppose that (i) holds. Then diagonal operators $\{H_a\}_{a\in\Gamma}$ exist so that XX, YY, HX, HY, and XY hold. The eigenvalue set of $\{H_a\}_{a\in\Gamma}$ is contained in $\{-1, 0, 1\}$. By restricting to $\{X_a, H_a\}_{a\in\Gamma}$, we know $\mathcal{FI}(P)$ carries a representation of $\mathfrak{b}'_+$. By Lemma 5.1.2 we know this representation is X-square nilpotent. By Proposition 3.1.1, we know the representation of $\mathfrak{b}'_-$ obtained by restricting to $\{Y_a, H_a\}_{a\in\Gamma}$ is Y-square nilpotent. Since XY(i) is the relation $[X_a, Y_a] = H_a$ for all $a \in \Gamma$, we see that (iii) holds.

Now suppose (iii) holds for some such diagonal operators $\{H_a\}_{a\in\Gamma}$. The relations XX, YY, HX, and HY are satisfied. Since the representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ are respectively X- and Y-square nilpotent, Proposition 3.1.1 shows P satisfies EC and ND. Lemma 5.2.1 and the assumed relation $[X_a, Y_a] = H_a$ for all $a \in \Gamma$ show XY holds. Thus $\mathcal{FI}(P)$ carries a representation of $\mathfrak{g}'$. Let $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$. By (iii) we have $H_b.\langle F, I\rangle = X_b Y_b.\langle F, I\rangle - Y_b X_b.\langle F, I\rangle$. First suppose b is the color of a minimal element y of F. With EC in mind, we have $Y_b X_b.\langle F, I\rangle = Y_b.\langle F - y, I + y\rangle = \langle F, I\rangle$. By EC and ND we see I has no maximal element of color b, so $X_b Y_b.\langle F, I\rangle = 0$. Thus $H_b.\langle F, I\rangle = -\langle F, I\rangle$. Dualize to show $H_b.\langle F, I\rangle = +\langle F, I\rangle$ if b is the color of a maximal element of I. If b is neither the color of a maximal element of I nor of a minimal element of F, then both $X_b Y_b.\langle F, I\rangle$ and $Y_b X_b.\langle F, I\rangle$ vanish. Thus $H_b.\langle F, I\rangle = 0$, so (iv) holds. This also shows that the eigenvalue set of the $\{H_a\}_{a\in\Gamma}$ is contained in $\{-1, 0, 1\}$. So (i) also holds.

Now suppose (iv) holds for some such diagonal operators $\{H_a\}_{a\in\Gamma}$. By Lemma 5.1.2 and its dual we know the representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ are X- and Y-square nilpotent, respectively. By Proposition 3.1.1 we know P satisfies EC and ND. Let $b \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$. To show $[X_b, Y_b] = H_b$, first suppose $Y_b X_b.\langle F, I\rangle \neq 0$. Then b is the color of a minimal element of F. With EC in mind we have $Y_b X_b.\langle F, I\rangle = \langle F, I\rangle$. By EC and ND we see I has no maximal element of color b, so $X_b Y_b.\langle F, I\rangle = 0$. Thus $[X_b, Y_b].\langle F, I\rangle = -\langle F, I\rangle = H_b.\langle F, I\rangle$, the last equality following from (iv). Next suppose $X_b Y_b.\langle F, I\rangle \neq 0$. Dualize the above to obtain $[X_b, Y_b].\langle F, I\rangle = +\langle F, I\rangle = H_b.\langle F, I\rangle$. Finally suppose $Y_b X_b.\langle F, I\rangle = 0$ and $X_b Y_b.\langle F, I\rangle = 0$. Then b is neither the color of a minimal element of F nor of a maximal element of I. Hence $[X_b, Y_b].\langle F, I\rangle = 0 = H_b.\langle F, I\rangle$, again using (iv) for the last equality. Thus (iii) holds.

Corollary 5.1.3 established the equivalence of (ii), (iv), and (v).

This proof showed that the diagonal operators $\{H_a\}_{a\in\Gamma}$ satisfying (iii) also satisfy the conditions of (iv). Hence the diagonal operators $\{H_a\}_{a\in\Gamma}$ for (i) also satisfy (iv). Corollary 5.1.3 showed that the μ-diagonal operators are the unique operators satisfying the conditions of (ii) or (iv). □

5.3 Upper P-minuscule representations of $\mathfrak{b}'_+$ in the general case

We begin to extend the results of Chapter 5 to the general case. See Section 2.5 for a description of numbering conventions and a summary of changes between the simply laced and general cases. Here we refer to the extended definitions for the general case made in Section 2.4 and the general case definitions and results of Sections 3.3, 3.4, 4.4, 4.5, and 4.6. The statement of our first main result remains the same except for changing "simple graph" to "Dynkin diagram" and changing I2A, Mx1GA, and Mn1LA to the respective extended properties I2∨1A, Mx1SB, and Mn1SB.

Theorem 5.3.1. *Let P be a poset whose elements are colored by the nodes of a Dynkin diagram Γ. Let $\mathcal{FI}(P)$ be the lattice of filter-ideal splits of P. The following are equivalent:*

(i) The lattice $\mathcal{FI}(P)$ carries an upper (respectively lower) P-minuscule representation of $\mathfrak{b}'_+$ (respectively $\mathfrak{b}'_-$).

(ii) The poset P satisfies EC, NA, AC, I2$\vee$1A, and Mx1SB (respectively Mn1SB).

When either of these conditions is satisfied, the μ-diagonal operators $\{M_a\}_{a\in\Gamma}$ of Section 4.4 can be used to give the actions of the $\{h_a\}_{a\in\Gamma}$.

The structure of this proof is the same as the proof of Theorem 5.1.1. However, the computational details change when using the extended functions $\{\mu_a\}_{a\in\Gamma}$ along with the updated frontier census bounds Mx1SB and Mn1SB. So for clarity, we completely rewrite the proof.

Proof. Assume that (i) holds for $\mathfrak{b}'_+$. Since this representation is X-square nilpotent, by Theorem 3.4.8 we know P satisfies EC and NA. Let $\{H_a\}_{a\in\Gamma}$ be the diagonal operators for this representation with $\mathfrak{h}'$-weight $\{\eta_a\}_{a\in\Gamma}$ and eigenvalue set $\mathcal{E}_\eta := \{\eta_a(F, I) \mid a \in \Gamma, (F, I) \in \mathcal{FI}(P)\}$. By Corollary 3.4.6(a) we know that $\{\eta_a\}_{a\in\Gamma}$ is a component weight function. Since the representation is upper P-minuscule, for all $a \in \Gamma$ and $(F, I) \in \mathcal{FI}(P)$:

$$\eta_a(F, I) \in \mathcal{E}_\eta \subseteq \{-1, 0, 1, 2, \dots\} \tag{5.3}$$

$$\eta_a(F, I) = -1 \text{ if and only if } a \text{ is the color of a minimal element of } F. \tag{5.4}$$

By (5.4) and Proposition 4.5.1 we see P satisfies AC and I2$\vee$1A. For the sake of contradiction, suppose Mx1SB fails. Then there is a color $b \in \Gamma$ and an element y maximal in P_b such that either there are infinitely many elements greater than y with colors adjacent to b, or there are finitely many such elements and $\sum_{z>y} -\theta_{\kappa(z),b} \geq 2$. In each case at least one of two possibilities must hold: There is an element $t > y$ with $-\theta_{\kappa(t),b} \geq 2$, or there are two elements u and v greater than y with $-\theta_{\kappa(u),b} - \theta_{\kappa(v),b} \geq 2$. Suppose there is an element $t > y$ with $-\theta_{\kappa(t),b} \geq 2$. Let I' be generated by t. Define $I := I' - (y, t]$. Note that I is an ideal of P and that y is maximal in I. Define $F' := P - I'$ and $F := P - I$. Further note that $I' - I = (y, t]$ and $I - I' = \emptyset$. Thus the splits (F', I') and (F, I) are in the same component of $\mathcal{FI}(P)$. Using Equation (3.6) we get

$$\eta_b(F', I') - \eta_b(F, I) = \sum_{c\in\Gamma} \theta_{cb} \Delta_c[(F', I'), (F, I)] = \sum_{c\in\Gamma} \theta_{cb} |P_c \cap (y, t]|.$$

Since y is maximal in P_b we have $|P_b \cap (y,t]| = 0$. Note that $\theta_{cb}|P_c \cap (y,t]| \leq 0$ for $c \neq b$. Thus since $-\theta_{\kappa(t),b} \geq 2$ we have $\sum_{c\in\Gamma} \theta_{cb}|P_c \cap (y,t]| \leq -2$. So we get the inequality $\eta_b(F',I') - \eta_b(F,I) = \sum_{c\in\Gamma} \theta_{cb}|P_c \cap (y,t]| \leq -2$. Since y is maximal in I and $\kappa(y) = b$, by the penultimate statement of Proposition 4.5.1 we get $\eta_b(F,I) = 1$. Thus the inequality becomes $\eta_b(F',I') \leq -1$. By (5.3) we have $\eta_b(F',I') \geq -1$, and so $\eta_b(F',I') = -1$. So by (5.4) we know that F' has a minimal element x of color b. Since $y \in I'$, by EC we have $y < x$. But y is maximal in P_b, so this is a contradiction. Otherwise there are two elements u and v greater than y with $-\theta_{\kappa(u),b} - \theta_{\kappa(v),b} \geq 2$. In this case let I' be the ideal generated by u and v, let $I := I' - ((y,u] \cup (y,v])$, and proceed with a similar argument. Thus Mx1SB must hold, so (ii) holds.

Now assume that (ii) holds. Since P satisfies EC, AC, and I2$\vee$1A, we know by Proposition 4.4.2 that $\{\mu_a\}_{a\in\Gamma}$ is a component weight function. We use the operators $\{M_a\}_{a\in\Gamma}$ specified by $\{\mu_a\}_{a\in\Gamma}$ to get the desired actions of the $\{h_a\}_{a\in\Gamma}$. Using EC, NA, and I3NE (which is implied by I2$\vee$1A), Corollary 3.4.6(b) says $\mathcal{FI}(P)$ carries an X-square nilpotent representation of $\mathfrak{b}'_+$ when using the $\{M_a\}_{a\in\Gamma}$. The eigenvalue set $\mathcal{E}_\mu$ of these operators is contained in $\mathbb{Z}$. Let $b \in \Gamma$ and $(F,I) \in \mathcal{FI}(P)$. By Corollary 4.6.3(a) we know that $\mu_b(F,I) \geq -1$, and so $\mathcal{E}_\mu \subseteq \{-1, 0, 1, 2, \dots\}$. The last statement of Proposition 4.5.1 says $\mu_b(F,I) = -1$ if b is the color of a minimal element of F.

To finish, we need to confirm the "only if" direction of Part (ii) of the definition of upper P-minuscule. So suppose $\mu_b(F,I) = -1$. Start with the case $P_b \cap I \neq \emptyset$. Then $-1 = \mu_b(F,I) = 1 - \upsilon_b(F,I)$, so $\upsilon_b(F,I) = 2$. This shows $P_b \cap I$ has a maximal element y since $\upsilon_b(F,I) = 1$ otherwise. Suppose there is some $c \in \Gamma$ with $c \sim b$ such that there are infinitely many elements greater than y in $P_c \cap I$. Then $P_b \cap F$ is empty by AC and local finiteness. But then y is maximal in P_b and Mx1SB is violated. Thus we get $\sum_{z\in\Upsilon_b(F,I)} -\theta_{\kappa(z),b} = \upsilon_b(F,I) = 2$. The elements in $\Upsilon_b(F,I)$ are greater than y, so we see by Mx1SB that y is not maximal in P_b. Let $x \in P_b$ be such that $y < x$ are consecutive occurrences of the color b. Since y is maximal in $P_b \cap I$, we have $x \in F$. Suppose $w \in P$ with $w \to x$. Then NA implies that $\kappa(w) \sim b$. Thus by AC we have $w \in (y,x)$. By I2$\vee$1A we know that $\sum_{z\in(y,x)} -\theta_{\kappa(z),b} = 2$, and so $\sum_{z\in(y,x)} -\theta_{\kappa(z),b} = \sum_{z\in\Upsilon_b(F,I)} -\theta_{\kappa(z),b}$. By AC we know that $\Upsilon_b(F,I) \subseteq (y,x)$. Thus $w \in \Upsilon_b(F,I)$ since otherwise we would have $\sum_{z\in(y,x)} -\theta_{\kappa(z),b} > \sum_{z\in\Upsilon_b(F,I)} -\theta_{\kappa(z),b}$. This shows that $w \in I$, so x is a minimal element of F with color b.

Otherwise we have the case $P_b \cap I = \emptyset$. Then $-1 = \mu_b(F,I) = -1 + \psi_b(F,I)$, so $\psi_b(F,I) = 0$. This shows $P_b \cap F$ has a minimal element y since $\psi_b(F,I) = 1$ otherwise. Suppose $w \in P$ with $w \to y$. Again by NA we know that $\kappa(w) \sim b$. If there is some color $c \sim b$ such that there are infinitely many elements less than y in $P_c \cap F$, then $\psi_b(F,I) \geq 1$. Since $\psi_b(F,I) = 0$, there are only finitely many elements less than y in $P_{\kappa(w)} \cap F$.

Any such elements would therefore be in $\Psi_b(F,I)$. But since $0 = \psi_b(F,I) = \sum_{z\in\Psi_b(F,I)} -\theta_{\kappa(z),b}$, there cannot be such elements. Thus $w \notin \Psi_b(F,I)$, so w must be in I. Hence y is minimal in F and has color b. Thus $\mu_b(F,I) = -1$ if and only if b is the color of a minimal element of F. Hence this representation of $\mathfrak{b}'_+$ carried by $\mathcal{FI}(P)$ is upper P-minuscule.

Dualize to obtain the analogous equivalence for $\mathfrak{b}'_-$. □

The statement of Lemma 5.1.2 is unchanged and the previous proof can be applied verbatim:

Lemma 5.3.2. *Let $\{H_a\}_{a\in\Gamma}$ be diagonal operators on V. Suppose the operators $\{X_a, H_a\}_{a\in\Gamma}$ satisfy HX. If the eigenvalue set of $\{H_a\}_{a\in\Gamma}$ is contained in $\{-1,0,1\}$, then the actions of the color raising operators $\{X_a\}_{a\in\Gamma}$ are X-square nilpotent.*

Corollary 5.1.3 also has the same statement and proof as before:

Corollary 5.3.3. *The Conditions (ii), (iv), and (v) of Theorem 5.4.2 are equivalent. When any of these conditions are satisfied, the μ-diagonal operators $\{M_a\}_{a\in\Gamma}$ are the unique operators satisfying (ii) or (iv).*

5.4 P-minuscule representations of $\mathfrak{g}'$ in the general case

Lemma 5.2.1 has the same statement and proof as before:

Lemma 5.4.1. *Suppose P satisfies EC and ND. The relation $[X_b, Y_a] = 0$ holds when $a, b \in \Gamma$ are distinct.*

Theorem 5.2.2 was our foremost main result. We update its statement by changing "simple graph" to "Dynkin diagram" and the properties I2A, Mx1GA, and Mn1LA to I2∨1A, Mx1SB, and Mn1SB respectively. Otherwise, the proof given in Section 5.2 can be applied verbatim.

Theorem 5.4.2. *Let P be a poset whose elements are colored by the nodes of a Dynkin diagram Γ. Let $\mathcal{FI}(P)$ be the lattice of filter-ideal splits of P. The following are equivalent:*

(i) The lattice $\mathcal{FI}(P)$ carries a P-minuscule representation of $\mathfrak{g}'$.

(ii) The lattice $\mathcal{FI}(P)$ carries upper and lower P-minuscule representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ respectively, using the same diagonal operators $\{H_a\}_{a\in\Gamma}$.

(iii) The lattice $\mathcal{FI}(P)$ carries X- and Y-square nilpotent representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ respectively, using diagonal operators $\{H_a\}_{a\in\Gamma}$ that satisfy $[X_a, Y_a] = H_a$ for every color $a \in \Gamma$.

(iv) The lattice $\mathcal{FI}(P)$ carries representations of $\mathfrak{b}'_+$ and $\mathfrak{b}'_-$ with diagonal operators $\{H_a\}_{a\in\Gamma}$ that satisfy for every $b \in \Gamma$ and every split (F,I):

$H_b.\langle F, I\rangle = -\langle F, I\rangle$ *if* b *is the color of a minimal element of* F*,*

$H_b.\langle F, I\rangle = +\langle F, I\rangle$ *if* b *is the color of a maximal element of* I*, and*

$H_b.\langle F, I\rangle = 0$ *otherwise.*

(v) The poset P *satisfies the properties EC, NA, AC, I2*$\vee$*1A, Mx1SB, and Mn1SB.*

When any of these conditions are satisfied, the μ*-diagonal operators* $\{M_a\}_{a\in\Gamma}$ *of Section 4.4 are the unique diagonal operators satisfying (i), (ii), (iii), or (iv).*

CHAPTER 6

The characterization

Here we summarize the characterizations of upper P-minuscule representations of $\mathfrak{b}'_+$ and P-minuscule representations of $\mathfrak{g}'$. For the remainder of this dissertation, we refer exclusively to the general case unless the simply laced assumption is explicitly made. We first give names to the posets characterizing these representations and then summarize these characterizations in Theorem 6.1.1. The historical motivation for our terminology comes from the close connections of our new posets to the d-complete [Pr3, Pr4, Pr5, Pr6] and minuscule [Pr1, Pr2] posets of Proctor. These connections will be made explicit in Chapters 7 and 8.

6.1 Γ-Colored d-complete and Γ-colored minuscule posets

In [Str] we defined a locally finite poset P (of any cardinality) colored with a finite simple graph Γ to be a Γ-*colored d-complete* poset if it satisfied EC, NA, AC, I2A, and Mx1GA with respect to Γ. Table 2.1 explains these abbreviations and indexes the locations of their definitions. We apply the same terminology to the general case, now using the generalizations of these properties: A locally finite poset P (of any cardinality) colored with a Dynkin diagram Γ is a Γ-*colored d-complete* poset if it satisfies EC, NA, AC, I2∨1A, and Mx1SB with respect to Γ. Table 2.4 explains the updated abbreviations and indexes the locations of the updated definitions. Figure 2.2 displayed a Γ-colored d-complete poset. Any nonempty filter of the poset displayed in Figure 2.1 is Γ-colored d-complete.

In [Str] we also said a locally finite poset P is a Γ-*colored minuscule* poset if it satisfies EC, NA, AC, I2A, Mx1GA, and Mn1LA with respect to a simple graph Γ. We again apply the same terminology to the general case: We say P is a Γ-*colored minuscule* poset if it satisfies EC, NA, AC, I2∨1A, Mx1SB, and Mn1SB with respect to a Dynkin diagram Γ. Figure 2.1 displayed a Γ-colored minuscule poset.

The definitions of upper P-minuscule representations of $\mathfrak{b}'_+$ and P-minuscule representations of $\mathfrak{g}'$ were made at the end of Section 2.3. The following statement summarizes our two main results, Theorems 5.3.1 and 5.4.2, which respectively characterized the upper P-minuscule and P-minuscule representations in the general case:

Theorem 6.1.1. *Let P be a poset whose elements are colored by the nodes of a Dynkin diagram Γ. Let $\mathcal{FI}(P)$ be the lattice of filter-ideal splits of P.*

(a) The lattice $\mathcal{FI}(P)$ carries an upper P-minuscule representation of $\mathfrak{b}'_+$ if and only if P is a Γ-colored d-complete poset.

(b) The lattice $\mathcal{FI}(P)$ carries a P-minuscule representation of $\mathfrak{g}'$ if and only if P is a Γ-colored minuscule poset.

In Section 9.1 we compare the "if" direction of Part (b) to Theorem 4.1.6(i) of [Gr3].

CHAPTER 7

Preparing for the classification

The remainder of this dissertation classifies the Γ-colored d-complete and Γ-colored minuscule posets, which were defined in Section 6.1. As a consequence of Theorem 6.1.1, this will respectively classify the upper P-minuscule representations of $\mathfrak{b}'_+$ and P-minuscule representations of $\mathfrak{g}'$. Our classification takes place in the setting of the general case. In this chapter we present what we need to prove the classification in Chapter 8. This will include the definitions of the classes of posets involved in the classification: Proctor's (colored) minuscule and d-complete posets, Stembridge's dominant minuscule heaps, and Green's full heaps. We dedicate one section each to presenting the settings of Proctor, Stembridge, and Green. In the fourth section, we provide equivalences between various subsets of the properties used by Stembridge and Green and subsets of our coloring properties. In the final section of this chapter, we state the equivalence between the colored d-complete posets of Proctor and the dominant minuscule heaps of Stembridge colored by simply laced Dynkin diagrams. We use it to describe the "slant sum" decomposition of Γ-colored d-complete posets.

7.1 The (colored) minuscule and (colored) d-complete posets of Proctor

The posets considered in this section are finite and are not assumed to be colored *a priori*. Refer to [Hum] for the theory of semisimple Lie algebras and their representations discussed here.

An irreducible highest weight representation of a semisimple Lie algebra $\mathfrak{g}$ with Dynkin diagram Γ is *minuscule* if every weight of the representation is in the Weyl group orbit of the highest weight. The highest weights for such representations are the *minuscule weights*. See [Bou, Ch. VIII, §7.3] and [Hum, Exer. 13.13] for more on the minuscule weights.

Given a minuscule representation of $\mathfrak{g}$, the weights of this representation form [Pr1, Prop. 4.1] a distributive lattice under the usual ordering on the dual space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$: For $\mu, \nu \in \mathfrak{h}^*$, we have $\mu \leq \nu$ if $\nu - \mu$ is a non-negative integral sum of simple roots. Following [Pr1, §4], here we define the (uncolored) *minuscule poset* associated to this representation to be the subposet of join irreducible elements of this distributive lattice. The connected minuscule posets were listed in Proposition 4.2 of [Pr1] and displayed in [Pr2, Fig. 2] and [McG, Appendix B]. Later in [Pr1], Theorem 11 assigned a simple root to each element of a minuscule poset; this assignment can be viewed as a Γ-coloring of the minuscule poset. We define the *colored minuscule posets* to be the resulting Γ-colored posets. In Table 7.1 we list the minuscule weights for the connected Dynkin diagrams from Exercise 13.13 of [Hum] and the names of the corresponding connected minuscule posets from [Pr1].

Lie Type	**Minuscule weight(s)**	**Minuscule poset(s)**	**Restriction(s)**
A_n	ω_j	$a_n(j)$	$n \geq 1$ and $1 \leq j \leq n$
B_n	ω_n	$b_n(n)$	$n \geq 2$
C_n	ω_1	$c_n(1)$	$n \geq 3$
D_n	ω_1 and ω_{n-1} and ω_n	$d_n(1)$ and $d_n(n-1)$ and $d_n(n)$	$n \geq 4$
E_6	ω_1 and ω_6	$e_6(1)$ and $e_6(6)$	N/A
E_7	ω_7	$e_7(7)$	N/A

Table 7.1: The list of minuscule weights and posets for semisimple $\mathfrak{g}$ with fundamental weights $\omega_1, \ldots, \omega_n$

Proctor later axiomatically defined the d-complete posets, which are generalizations of the minuscule posets. There are several equivalent definitions of d-complete posets. The first version of this definition to appear in the literature was given in [Pr3, Pr4]. The definitions given in [Pr4] were dualized; here we will implicitly undo that dualization when citing [Pr4]. The definition of d-complete we present in this section first appeared in [Pr6]. It was later restated as as Definition 9.1 of [PrSc]. The definition of d-complete given in [Pr3, Pr4] was referred to as the "Classic" definition in [PrSc] and was presented in that paper in Table 9.1(a). Theorem 9.2 of that paper showed the definition we use here [Pr6] is equivalent to the Classic definition of [Pr3, Pr4].

Let $k \geq 3$. The minuscule poset $d_k(1)$ is the poset consisting of $2k-2$ elements, where exactly two are incomparable and exactly $k-2$ are greater than each of those two incomparable elements; these latter elements are the *neck elements*. The poset $d_k(1)^-$ is the poset produced by removing the maximal element from $d_k(1)$. A *double-tailed diamond* is any poset isomorphic to $d_h(1)$ for some $h \geq 3$. A *d_k-interval* of a poset P is an interval isomorphic to $d_k(1)$. A *d_k^--convex set* is a convex subset $S \subseteq P$ that is isomorphic to $d_k(1)^-$. The poset P is *d-complete* if for every $h \geq 3$: Whenever S is a d_h^--convex set in P, there exists an element of P which covers exactly the maximal element(s) of S and which does not cover the maximal element(s) of any other d_h^--convex set.

Proctor classified [Pr3] the d-complete posets. To do this, he showed that every connected d-complete poset can be decomposed as a "slant sum" of "slant irreducible" d-complete posets, and then he classified the slant irreducible d-complete posets. There are fifteen such families of slant irreducible d-complete posets. This slant sum decomposition is described below in Section 7.5.

We now present the axiomatic definition of colored d-complete posets introduced in [Pr4]. Once again let P be any poset. Let $\kappa : P \to \Gamma$ be a coloring map to a *set* of colors Γ. Here we do not require Γ to have a graph structure *a priori*. Proctor defined P to be *properly colored* if no two incomparable elements are colored the same (i.e. the poset P satisfies EC) and no element is colored the same as an element it covers (i.e. the poset P satisfies ND). Proctor defined P to be *simply colored* if it is properly colored and whenever an interval is a chain, the colors of the elements in that interval are distinct. (This implies I3ND.) Finally, the poset P was defined to be a *colored d-complete poset* if it is a d-complete and simply colored poset satisfying for all $x, y \in P$: Whenever $[x, y]$ is a d_k-interval for some $k \geq 3$, we have $\kappa(x) = \kappa(y)$. In Fact 7.5.1 below Γ will be given the structure of a simply laced Dynkin diagram. We note that connected d-complete posets have a unique maximal element [Pr3, PrSc].

Checking case-by-case, it is easy to see that the minuscule posets are uncolored d-complete posets. It can also be checked that the node assignments of Theorem 11 of [Pr1] satisfy the d-complete coloring requirements of [Pr4] in the simply laced cases; this result is due to Proctor (personal communication):

Fact 7.1.1. *Suppose Γ is simply laced. A Γ-colored poset is a colored minuscule poset (as defined above) if and only if as an uncolored poset it is isomorphic to a minuscule poset and its coloring satisfies the d-complete coloring requirements of [Pr4].*

Each of these posets has an associated Dynkin diagram Γ. However, here we do not incorporate "Γ-" into the terminology "colored minuscule poset"; refraining from doing so makes this terminology distinct from the terminology of "Γ-colored minuscule" for the posets of Section 6.1 that are defined with our six coloring properties.

7.2 The dominant minuscule heaps of Stembridge

We continue to assume that all posets are finite in this section. Stembridge reformulated [Ste] the colored d-complete axioms of Proctor into a set of coloring axioms that became the inspiration for the full heap axioms of Green [Gr3]. Let P be a poset colored by a Dynkin diagram Γ (as in Section 2.4) with coloring map $\kappa : P \to \Gamma$. Stembridge did not require Γ to be simply laced, nor did he require κ to be surjective. We continue to require κ to be surjective. For adjacent colors $a, b \in \Gamma$, Stembridge defined b to be *short relative to a* if $\theta_{ba} = -1$. Clearly b is short relative to a precisely when it is 1-adjacent to a, so we will continue to use our adjacency terminology when recalling Stembridge's axioms. To compare Stembridge's axioms to our properties, we introduce a relaxed version of NA:

(NWA): Neighbors have weakly adjacent colors.

Stembridge defined the poset P to be a *dominant minuscule heap* if it satisfies the following properties:

(S1): All covering pairs in P have colors that are equal or adjacent in the Dynkin diagram Γ, and incomparable pairs have distinct, non-adjacent colors.

(S2): Every open subinterval of P between two consecutive elements colored a contains either

(i) exactly two elements whose colors are adjacent to a, and both colors are 1-adjacent to a, or

(ii) exactly one element, and the label of this element, say b, satisfies $\theta_{ba} = -2$.

(S3): An element that is maximal in P among all elements colored a is covered by at most one element, and this element is maximal among all elements of some color that is 1-adjacent to a.

(S4): The colors that occur in P index an acyclic subset of the Dynkin diagram.

Since we still require κ to be surjective, the property S4 requires Γ to be acyclic. The poset P was said to be a *minuscule heap* if it satisfies S1 and S2. His S1 is the conjunction of our EC, NWA, and AC. His S2 implies our I2$\vee$1A, and the converse is true when NWA is present. So his minuscule heaps are our finite posets satisfying EC, NWA, and AC. Finally, his dominant minuscule heaps turn out to be our finite Γ-colored d-complete posets. These claims will be proved in Section 7.4.

Stembridge classified [Ste] the dominant minuscule heaps. To do this, he showed that every dominant minuscule heap can be decomposed into (using Proctor's language) a "slant sum" of "slant irreducible" dominant minuscule heaps, and then he classified the slant irreducible dominant minuscule heaps. His classification implicitly used the equivalence between the dominant minuscule heaps colored by simply laced Dynkin diagrams and the colored d-complete posets, which we state in Fact 7.5.1. After applying this equivalence and using Proctor's classification [Pr3] of the fifteen families of "slant irreducible" d-complete posets, he finished the classification by introducing two additional families of slant irreducible dominant minuscule heaps colored by multiply laced Dynkin diagrams. We present a detailed description of this "slant sum" decomposition in Proposition 7.5.3.

7.3 The full heaps of Green

We no longer assume that all posets are finite. Using Stembridge's coloring axioms as a starting point, Green developed a new type of "doubly infinite" colored poset called a "full heap" that is closely related to the notion of Γ-colored minuscule poset. We give the most recent definition of full heap [Gr3]; earlier versions appeared in [Gr1, Gr2]. Let P be a poset colored by any Dynkin diagram Γ with coloring map $\kappa : P \to \Gamma$. Green did not require Γ to have finitely many nodes, and when defining "heap" (given by G1 and G2 below) he did not require P to be locally finite. He also did not require κ to be surjective. (The surjectivity of κ was obtained *a posteriori* in [Gr3] using the full heap axiom G3 below.) To be consistent with the rest of this dissertation, we use the transpose of Green's convention for the entries of the generalized Cartan matrix.

Green defined a poset P to be a *heap* if it satisfies the following two properties:

(G1): For every $a \in \Gamma$ and every edge $\{b, c\}$ of Γ, the subsets $\kappa^{-1}(a)$ and $\kappa^{-1}(\{b, c\})$ of P are chains with respect to the partial order on P.

(G2): The partial order on P is the minimal partial order extending the given partial order on the above chains $\kappa^{-1}(a)$ and $\kappa^{-1}(\{b, c\})$.

For $a \in \Gamma$ and edge $\{b, c\}$ in Γ, we call $\kappa^{-1}(a)$ a *color chain* and $\kappa^{-1}(\{b, c\})$ an *edge chain*. Note that $\kappa^{-1}(a)$ in our notation is P_a. Green then defined P to be a *full heap* if it is a locally finite heap that also satisfies:

(G3): For every $a \in \Gamma$, the set P_a is isomorphic as a poset to $\mathbb{Z}$.

(G4): Whenever $a, b \in \Gamma$ are adjacent and $x \in P_a$, there is an element $y \in P_b$ such that $x \to y$ or $y \to x$.

(G5): For all $a \in \Gamma$: If $x < y$ are consecutive occurrences of the color a, then $\sum_{z \in [x,y]} \theta_{\kappa(z),a} = 2$.

Whenever we recall or assume Green's axioms it will be in our standard setting of a locally finite poset surjectively colored by a Dynkin diagram with finitely many nodes. In this setting Green's axioms G1 and G2 together are equivalent to our properties EC, NWA, and AC. His axiom G5 is equivalent to our I2$\vee$1A. These claims will be proved in Section 7.4.

It turns out that G4 was in fact implied by Green's other axioms:

Proposition 7.3.1. *Let P be a poset colored by a Dynkin diagram Γ with coloring map $\kappa : P \to \Gamma$. If P satisfies G1, G2, G3, and G5, then it also satisfies* G4*. Hence G4 can be removed from the definition of full heap.*

Proof. Suppose P fails G4. Then there is a pair of adjacent colors $a, b \in \Gamma$ and an element $x \in P_a$ that neither covers nor is covered by any element of P_b. By G3 the set P_b is isomorphic to $\mathbb{Z}$. By G1 all elements of P_b are comparable to x. By local finiteness, we may choose elements $u, v \in P_b$ such that u is the maximal element of color b less than x and v is the minimal element of color b greater than x. Then $u < v$ are consecutive elements of the color b. Since P is locally finite we may choose a saturated chain $u \to y \to \cdots \to x \to \cdots \to z \to v$ from u to v in P. Since x is not a neighbor of any element of color b, we know $y \neq x \neq z$. By G2 all covering pairs in P have equal or adjacent colors in Γ. But by G5 the colors of covering pairs cannot be equal. So $b \sim \kappa(y)$ and $b \sim \kappa(z)$. By assumption we also have $b \sim a = \kappa(x)$. Hence $\theta_{\kappa(y),b} \leq -1$ and $\theta_{\kappa(z),b} \leq -1$ and $\theta_{\kappa(x),b} \leq -1$. The only positive numbers in the sum $\sum_{w \in [u,v]} \theta_{\kappa(w),b}$ are $\theta_{\kappa(u),b} = \theta_{\kappa(v),b} = 2$. Thus $\sum_{w \in [u,v]} \theta_{\kappa(w),b} < 2$, contradicting G5. □

The full heaps over Dynkin diagrams of affine type were classified by Green in Theorem 6.6.2 of [Gr3]. Green's student McGregor-Dorsey later showed in Theorem 4.7.1 of [McG] that if P is a full heap over a Dynkin diagram Γ with finitely many nodes, then Γ consists of connected components of affine type. Thus for *any* connected Γ, a full heap P for Γ already appeared in the list in Theorem 6.6.2 of [Gr3]. We will see in Corollary 8.1.3 that the list in this theorem becomes the complete list of connected full heaps, since that result implies that a full heap is connected if and only if its Dynkin diagram is connected.

7.4 Equivalences between sets of our properties and sets from Stembridge and Green

In this section we continue to assume that P is a locally finite poset colored by a Dynkin diagram Γ with a surjective coloring map $\kappa : P \to \Gamma$. Various subsets of the properties used to define the dominant minuscule heaps, the full heaps, and our Γ-colored minuscule posets are equivalent. Recall that a poset satisfying G1 and G2 is a heap.

Proposition 7.4.1. *Let P be a poset colored by a Dynkin diagram Γ with coloring map $\kappa : P \to \Gamma$. The following are equivalent:*

(i) The poset P satisfies S1.

(ii) The poset P satisfies G1 and G2.

(iii) The poset P satisfies EC, NWA, and AC.

Proof. The property EC can be restated to say that incomparable elements have distinct colors. The property AC can be restated to say that incomparable elements have non-adjacent colors. The property NWA can be restated to say that covering pairs have colors that are equal or adjacent in Γ. Thus EC, AC, and NWA together are precisely S1, so (i) and (iii) are equivalent.

Suppose that (ii) holds. The property G1 clearly implies EC and AC. By G2, the only covering relations in P are those coming from the color and edge chains. Thus all neighbors in P have colors that are equal or adjacent in Γ, so NWA holds. Now suppose that (iii) holds. The properties EC and AC clearly imply G1. Let $x, y \in P$ be such that $x < y$. Since P is locally finite, choose a finite saturated chain $x \to \cdots \to y$ in P. By NWA, the edges along this saturated chain must belong to color chains or edge chains. Hence G2 holds. $\square$

The next result addresses the subtle difference between I2∨1A and S2 that was mentioned in Section 4.4.

Proposition 7.4.2. *Let P be a poset colored by a Dynkin diagram Γ with coloring map $\kappa : P \to \Gamma$. The following are equivalent:*

(i) The poset P satisfies G5.

(ii) The poset P satisfies I2∨1A.

If P additionally satisfies NWA, then (i) and (ii) are each also equivalent to:

(iii) The poset P satisfies S2.

Proof. Suppose (i) holds. Let $a \in \Gamma$ and let $x < y$ be consecutive elements of the color a. By G5 we know that $\sum_{z\in[x,y]} \theta_{\kappa(z),a} = 2$. Since $\theta_{aa} = 2$ and x and y are the only elements in $[x, y]$ with color a, we have $\sum_{z\in(x,y)} \theta_{\kappa(z),a} = -2$. For all $z \in (x, y)$, the number $\theta_{\kappa(z),a}$ is a non-positive integer. If the sum is $-1 - 1$, then there are exactly two elements in (x, y) whose colors are adjacent to a. These colors are both 1-adjacent to a, so I2∨1A(i) holds. If the sum is a single term -2, then there is exactly one element in (x, y) whose color is adjacent to a. This color is 2-adjacent to a, so I2∨1A(ii) holds. Thus (ii) holds. Each step in this argument can be reversed, so (ii) implies (i) as well.

The property S2 clearly implies I2∨1A. We only need NWA for the converse: Assume NWA and suppose I2∨1A holds. Let $a \in \Gamma$ and suppose $x < y$ are consecutive occurrences of the color a. Note that I2∨1A(i) and S2(i) are identical. Otherwise, suppose I2∨1A(ii) applies. Let z be the unique element in (x, y) whose color $b := \kappa(z)$ is adjacent to a. By NWA, any element covering x must have a color that is equal or adjacent to a. Note y cannot cover x since $z \in (x, y)$. Thus z is the only element in (x, y) that can cover x, so we have $x \to z$. Similarly, we see z is the only element in (x, y) that can be covered by y, so $z \to y$. Thus z is the only element in (x, y), so S2(ii) holds for this interval. $\square$

We note that NA implies NWA. We also note that in the presence of S2, G5, or I2∨1A, the property NWA becomes NA. Recall that a finite poset satisfying S1 and S2 is a minuscule heap. Combining Propositions

7.4.1 and 7.4.2, we can translate this combination of properties into the other two settings at hand while more generally considering locally finite posets:

Corollary 7.4.3. *Let P be a poset colored by a Dynkin diagram Γ with coloring map $\kappa : P \to \Gamma$. The following are equivalent:*

(i) The poset P satisfies S1 and S2.

(ii) The poset P satisfies G1, G2, and G5.

(iii) The poset P satisfies EC, NA, AC, and I2∨1A.

Proof. First suppose P satisfies S1 and S2. By Proposition 7.4.1, we see that P satisfies G1, G2, and NWA. Hence by Proposition 7.4.2, we see that P also satisfies G5.

Now suppose P satisfies G1, G2, and G5. By Proposition 7.4.1, we see that P satisfies EC, AC, and NWA. By Proposition 7.4.2, we see that P satisfies I2∨1A. Recall that NWA becomes NA in the presence of I2∨1A.

Now suppose P satisfies EC, NA, AC, and I2∨1A. Since NA implies NWA, Proposition 7.4.1 implies that P satisfies S1 and Proposition 7.4.2 implies P satisfies S2. □

Combining Proposition 7.3.1 and Corollary 7.4.3 translates the notion of full heap into our setting:

Corollary 7.4.4. *Let P be a poset colored by a Dynkin diagram Γ with coloring map $\kappa : P \to \Gamma$. The following are equivalent:*

(i) The poset P is a full heap.

(ii) The poset P satisfies EC, NA, AC, I2∨1A, and G3.

The next result shows that when P is finite, the property Mx1SB reformulates Stembridge's S3 and S4 in the presence of one or two common conditions.

Proposition 7.4.5. *Let P be a finite poset colored by a Dynkin diagram Γ with coloring map $\kappa : P \to \Gamma$. Suppose P satisfies NA.*

(a) If P satisfies S3 and S4, then P satisfies Mx1SB.

(b) If P satisfies Mx1SB and also AC, then P satisfies S3 and S4.

Proof. To show (a), assume P satisfies S3 and S4 and suppose P fails Mx1SB. Then there is a color $a \in \Gamma$ and an element x that is maximal in P_a such that $\sum_{y>x} -\theta_{\kappa(y),a} > 1$. Let F be the filter generated by x. By repeatedly applying S3 if necessary, we see that F is a chain with no repeated colors. Since $\sum_{y>x} -\theta_{\kappa(y),a} > 1$, there is at least one element greater than x with color adjacent to a. Let z be maximal among such elements. If z covers x, then it is the only such element greater than x with color adjacent to a. This implies that $-\theta_{\kappa(z),a} > 1$, which violates S3. Thus z does not cover x. Let $y \in P$ cover x and let $x \to y \to \cdots \to z$ be a saturated chain from x to z. By NA this chain induces a path in Γ from a to b. Since F has no repeated colors, each color in this path is passed through only once. Since $x \to y \to \cdots \to z$ contains at least two pairs of neighbors, the path from a to b contains at least two edges; neither of these edges is the edge between a and b. But since $a \sim b$, there exists a cycle in Γ. This contradicts S4, and so P must satisfy Mx1SB.

To show (b), suppose P satisfies Mx1SB and AC. Let $a \in \Gamma$ and suppose x is maximal in P_a. Suppose x is covered by two elements y and z. By NA we have $\kappa(y) \sim a$ and $\kappa(z) \sim a$. Thus $\sum_{w>x} -\theta_{\kappa(w),a} > 1$, contradicting Mx1SB. Hence x is covered by at most one element. Suppose x is covered by an element y with $b := \kappa(y)$. We again see by NA that $b \sim a$. If y is not maximal in P_b, then by AC we see that $\sum_{w>x} -\theta_{\kappa(w),a} > 1$. If b is k-adjacent to a for some $k > 1$, then $\sum_{w>x} -\theta_{\kappa(w),a} > 1$. In either case Mx1SB is violated. Thus P satisfies S3. Now suppose Γ contains some cycle. Fix one and choose a linear extension $\mathcal{L}^*$ of the order dual poset P^*. Let x be the first occurrence in $\mathcal{L}^*$ of the last color a to appear from the cycle. Then x is the maximal element of color a in P. Let u and v be elements with colors adjacent to a appearing earlier in $\mathcal{L}^*$. Such elements must exist since a is adjacent to two colors in the cycle and κ is surjective. Then $x < u$ and $x < v$ in P by AC. Hence $\sum_{w>x} -\theta_{\kappa(w),a} > 1$, which violates Mx1SB. Thus P satisfies S4. □

Combining Corollary 7.4.3 and Proposition 7.4.5 gives the equivalence between Stembridge's dominant minuscule heaps and our finite Γ-colored d-complete posets:

Corollary 7.4.6. *Let P be a finite poset colored by a Dynkin diagram Γ with coloring map $\kappa : P \to \Gamma$. The following are equivalent:*

(i) The poset P is a dominant minuscule heap.

(ii) The poset P is a Γ-colored d-complete poset.

7.5 Decomposition of finite Γ-colored d-complete posets

Given Corollary 7.4.6, in this section we use the terminology "finite Γ-colored d-complete poset" and "dominant minuscule heap" interchangeably. When Γ is simply laced, the colored d-complete posets of [Pr4] could be included in Corollary 7.4.6 provided that the provision is made for the specification of Γ using the notion of "top tree." Proctor has proved (personal communication) this result, which is stated in next; its proof will appear in a forthcoming paper. Here we use Stembridge's definition of the top tree: The *top tree* T of a finite connected Γ-colored poset P is the subposet of maximal elements of each color.

Fact 7.5.1. *Let P be a finite connected poset.*

(a) Suppose P is a Γ-colored d-complete poset for a simply laced Dynkin diagram Γ. Then P is a colored d-complete poset.

(b) Suppose P is a colored d-complete poset with color set Γ and top tree T. Then T is a filter of P. Regard Γ as a colored simple graph by creating edges between colors that correspond to covering relations in T. Then P is a Γ-colored d-complete poset.

We continue to assume that P is connected. Proctor defined [Pr6] the top tree to be the set of all $x \in P$ such that the principal filter generated by x is a chain. Clearly this is a filter of P. Since connected colored d-complete posets have a unique maximal element, this is equivalent to the definition of top tree given in [Pr4]; the top tree was defined there as the set of all $x \in P$ such that whenever y is a maximal element of P, the set $[x, y]$ is a (possibly empty) chain. Proctor showed [Pr4, Prop. 8.6] that a d-complete poset can be colored uniquely with the colors appearing in its top tree (his definition), which were the maximum appearances of the colors. Thus for connected dominant minuscule heaps and colored d-complete posets the notions of top tree are equivalent; we continue to use Stembridge's definition.

We now describe the slant sum decomposition of the dominant minuscule heaps. Let P be a connected dominant minuscule heap over a Dynkin diagram Γ, and denote its top tree by T. Stembridge defined P to be *slant irreducible* if whenever $x, y \in T$ and $x \to y$, the element y is not the only element of its color in P. When Γ is simply laced, in [Pr3] Proctor defined P to be *slant irreducible* if whenever $x, y \in T$ and $x \to y$, the element y is not in the neck of any d_k-interval for any $k \geq 3$. Proctor's definition was made for uncolored d-complete posets in [Pr3], but it also makes sense in the context of [Pr4]. The next result shows that these two notions of slant irreducible are equivalent when Γ is simply laced.

Proposition 7.5.2. *Let P be a finite Γ-colored d-complete poset for some simply laced Dynkin diagram Γ. Let T be the top tree of P. Let $x, y \in T$ and let $x \to y$. The following are equivalent:*

(i) The element y is not in the neck of any d_k-interval for any $k \geq 3$.

(ii) The element y is the only element of its color in P.

Proof. First suppose (ii) fails. Since $y \in T$, it is the maximal element of color $b := \kappa(y)$ in P. Let $z < y$ be consecutive elements of the color b in P. Since Γ is simply laced, by [Ste, Prop. 3.3] we know $[z, y]$ is a

d_k-interval for some $k \geq 3$. Since y is a neck element of this interval, we see (i) fails.

Now suppose (i) fails. Let $z < y$ be such that $[z, y]$ is a d_k-interval for some $k \geq 3$. Since P is colored d-complete by Fact 7.5.1, we must have $\kappa(z) = \kappa(y)$. Thus (ii) fails. □

We call an edge $x \to y$ in T a *slant edge* if y is the only element of its color in P. We now describe the slant sum decomposition into slant irreducible pieces. This was obtained in [Pr3, §4] in the simply laced case and in [Ste, §4] in the general case (taking Fact 7.5.1 for granted); see [Ste] for the proof. Stembridge required the generalized Cartan matrix to be symmetrizable in order to make use of a symmetric bilinear form when characterizing [Ste, §3] the heaps of λ-minuscule Weyl group elements. Once he had obtained this axiomatic characterization, his classification [Ste, §4] of the dominant minuscule heaps used only the axioms S1–S4 and did not refer to the bilinear form.

Proposition 7.5.3. *Let P be a connected finite Γ-colored d-complete poset. Let T be the top tree of P and let l be the number of slant edges in T. Set $m := l + 1$. Then there are slant irreducible dominant minuscule heaps $P_1, \ldots, P_m$ respectively colored by Dynkin diagrams $\Gamma_1, \ldots, \Gamma_m$ with respective top trees $T_1, \ldots, T_m$ such that:*

(a) The top trees $T_1, \ldots, T_m$ are formed by removing from T exactly the l slant edges,

(b) The Dynkin diagrams $\Gamma_1, \ldots, \Gamma_m$ are formed by removing the corresponding edges from Γ, and

(c) The posets $P_1, \ldots, P_m$ are precisely the portions of P respectively colored by $\Gamma_1, \ldots, \Gamma_m$.

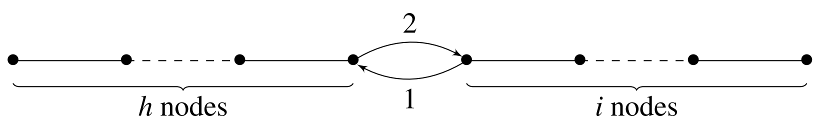
For each $1 \leq k \leq m$: If Γ_k is simply laced, then P_k belongs to one of the fifteen families of slant irreducible d-complete posets found in [Pr3]. If Γ_k is multiply laced, then P_k belongs to one of the two families of slant irreducible dominant minuscule heaps found in [Ste].

Here we say P is the *slant sum* of $P_1, \ldots, P_m$. In the simply laced case, the Hasse diagrams of the top trees $T_1, \ldots, T_m$ are the Dynkin diagrams $\Gamma_1, \ldots, \Gamma_m$ when they are viewed as graphs.

Let P be a finite Γ-colored d-complete poset. We now indicate how the structure of P constrains the values of θ_{cd} for $c, d \in \Gamma$. The Dynkin diagram Γ was defined to encode these values. Recall that $\theta_{ab} = \theta_{ba} = 0$ if and only if a and b are distant in Γ. We know by EC that two colors are distant whenever they color elements in different connected components of P, so assume P is connected. Let T be the top tree of P. Let $x, y \in T$ and set $a := \kappa(x)$ and $b := \kappa(y)$. Suppose $x \to y$ is a slant edge of T. In Proposition 7.5.3 the edge $x \to y$ is removed when computing the slant sum decomposition of P. Can we conclude anything about the nature of the adjacency in Γ between a and b guaranteed by NA? Since $x \in T$ is the maximal element of color a, we must have $-\theta_{ba} = 1$ by Mx1SB. Thus b is 1-adjacent to a. However, the color a may be k-adjacent to b for *any* $k \geq 1$. To see this, note that the properties EC, NA, and AC do not restrict the integers $\{\theta_{cd}\}_{c,d\in\Gamma}$, the property I2∨1A does not restrict θ_{ab} since y is the only element of color b in P, and Mx1SB does not restrict θ_{ab} since there is no element greater than y with color a. When combined with the classifications of Proctor [Pr3] and Stembridge [Ste], this discussion describes the Dynkin diagram structure of any finite Γ-colored d-complete poset. For connected such posets we have:

Corollary 7.5.4. *Let P be a finite connected Γ-colored d-complete poset with top tree T. Fix the integers l and $m := l + 1$ from Proposition 7.5.3. Let $\Gamma_1, \ldots, \Gamma_m$ be the Dynkin diagrams from Proposition 7.5.3(b).*

(a) Fix some $1 \leq k \leq m$. Then the diagram Γ_k is either simply laced and described by Table 1 of [Pr3], or it is multiply laced and has the form



for some $h, i \geq 1$.

Now suppose $l \geq 1$. Let $x_1 \to y_1, \ldots, x_l \to y_l$ be the slant edges in T from Proposition 7.5.3. For $1 \leq j \leq l$, set $a_j := \kappa(x_j)$ and $b_j := \kappa(y_j)$.

(b) For every $1 \leq j \leq l$, we have $\theta_{b_j,a_j} = -1$.

(c) For every $1 \leq j \leq l$, the positive integer $-\theta_{a_j,b_j}$ decorates the edge from a_j to b_j in Γ. Let Γ' be a Dynkin diagram created by replacing these l positive integers with any choice of l positive integers. Then P is a Γ'-colored d-complete poset.

Proof. Part (a) follows from the classification in [Pr3] and from Theorem 4.2 of [Ste]. Parts (b) and (c) follow from the discussion preceding the corollary. □

CHAPTER 8

The classification

We now classify the Γ-colored d-complete and Γ-colored minuscule posets. These definitions were introduced in Section 6.1 for locally finite posets P colored by Dynkin diagrams Γ with finitely many nodes. We first show that every Γ-colored d-complete poset P can be written as a disjoint union of connected Γ-colored d-complete posets, so it is enough to assume P is connected. The same statements are true for Γ-colored minuscule posets. Theorem 8.3.8 gives the classification of connected Γ-colored d-complete posets and Theorem 8.4.5 gives the classification of connected Γ-colored minuscule posets. The work leading to Theorem 8.3.8 obtains and then applies the frontier census property Mn2SB for Γ-colored d-complete posets; this is the most notable application of a frontier census property other than Mx1SB and Mn1SB. Theorem 8.5.1 is the culmination of this dissertation: It is the complete classification of upper P-minuscule representations of $\mathfrak{b}_+'$ and P-minuscule representations of $\mathfrak{g}'$.

8.1 Reducing the classifications to the connected cases

Our first result shows that the coloring properties used in this dissertation each respect direct sums of posets and disjoint unions of Dynkin diagrams. These properties are summarized in Tables 2.1 and 2.4, which also indexes their locations.

Lemma 8.1.1. *Let Γ_1 and Γ_2 be disjoint Dynkin diagrams and let P_1 and P_2 respectively be disjoint Γ_1- and Γ_2-colored posets. Set $\Gamma := \Gamma_1 \cup \Gamma_2$ and $P := P_1 + P_2$. Then each of the properties in Tables 2.1 and 2.4 hold for P if and only if they hold for P_1 and P_2. Hence P is Γ-colored d-complete (respectively Γ-colored minuscule) if and only if P_1 and P_2 are respectively Γ_1- and Γ_2-colored d-complete (respectively Γ_1- and Γ_2-colored minuscule).*

Proof. The "if" statement is obvious. Now assume P is Γ-colored d-complete. Each property in Tables 2.1 and 2.4 is stated in terms of comparable elements in the poset and equal or adjacent colors in the Dynkin diagram. Thus each property is preserved under this decomposition. □

The next result describes the direct sum decomposition of a Γ-colored poset when some common conditions are present:

Proposition 8.1.2. *Suppose P satisfies EC.*

(a) Then P is a disjoint union of at most $|\Gamma|$ connected posets; call them $Q_1, \ldots, Q_r$ for some $1 \leq r \leq |\Gamma|$.

(b) Additionally assume that P satisfies NA and AC. Then Γ has connected components $\Gamma_1, \ldots, \Gamma_r$ such that $\kappa(Q_k) = \Gamma_k$ for $1 \leq k \leq r$.

Proof. Elements in different connected components of P are incomparable. Taking one element from each connected component forms an antichain in P. By EC all elements of this antichain have different colors. Hence P must have at most $|\Gamma|$ connected components $Q_1, \ldots, Q_r$, so (a) holds. To prove (b), for every $1 \leq k \leq r$ define the set $\Gamma_k := \kappa(Q_k)$. Since κ is surjective we see that $\Gamma = \bigcup_{k=1}^{r} \Gamma_k$. If $x \in Q_i$ and $y \in Q_j$ for $1 \leq i, j \leq r$ such that $i \neq j$, then x and y have different colors by EC. Thus $\Gamma_i \cap \Gamma_j = \emptyset$. Hence the union of sets $\bigcup_{k=1}^{r} \Gamma_k$ is disjoint. Now make Γ_k a graph for all $1 \leq k \leq r$: The edges of Γ_k are the edges of Γ between the nodes of Γ_k. Continue to require $i \neq j$ and suppose there are adjacent colors $a \sim b$ in Γ with $a \in \Gamma_i$ and $b \in \Gamma_j$. Since κ is surjective, choose elements x and y respectively having colors a and b. By AC we know x and y are comparable in P. This is a contradiction since $x \in Q_i$ and $y \in Q_j$ are in different connected components of P. Hence the edge set of Γ is the disjoint union of the edge sets of $\Gamma_1, \ldots, \Gamma_r$. So Γ is the disjoint union of the graphs $\Gamma_1, \ldots, \Gamma_r$. The graphs $\Gamma_1, \ldots, \Gamma_r$ are Dynkin diagrams since the positive integers decorating the directed edges of Γ also decorate the directed edges of $\Gamma_1, \ldots, \Gamma_r$. It remains to show Γ_k is a connected graph for each $1 \leq k \leq r$. Fix $1 \leq l \leq r$ and let $a, b \in \Gamma_l$. Choose $x, y \in Q_l$ with $\kappa(x) = a$ and $\kappa(y) = b$. Since Q_l is connected, there is a finite path from x to y in its Hasse diagram. By NA, this path induces a path in Γ_l from a to b. Thus Γ_l is connected and (b) holds. $\square$

We get the following useful corollary:

Corollary 8.1.3. *Suppose P satisfies EC, NA, and AC. Then P is connected if and only if Γ is connected.*

The coloring properties for Γ-colored d-complete and Γ-colored minuscule posets respect the coloring decomposition of Proposition 8.1.2 by applying Lemma 8.1.1 a total of $r - 1$ times. Thus we get:

Corollary 8.1.4. *Suppose P satisfies EC, NA, and AC. Write P as a direct sum of connected posets $Q_1, \ldots, Q_r$ that are respectively colored by connected Dynkin diagrams $\Gamma_1, \ldots, \Gamma_r$ as in Proposition 8.1.2.*

(a) The poset P is Γ-colored d-complete if and only if Q_k is Γ_k-colored d-complete for $1 \leq k \leq r$.

(b) The poset P is Γ-colored minuscule if and only if Q_k is Γ_k-colored minuscule for $1 \leq k \leq r$.

So it is sufficient to classify the connected Γ-colored d-complete and Γ-colored minuscule posets.

Now suppose P is a finite Γ-colored d-complete poset. Corollary 8.1.4(a) says that P is a direct sum of finite connected Γ-colored d-complete posets. Corollary 7.5.4 can be used to describe the possibilities for the integers θ_{ab} for colors in the same connected component of Γ. Note that if c and d are colors in different connected components of Γ, then $c \nsim d$. Hence we get $\theta_{cd} = \theta_{dc} = 0$. This describes all possibilities for the integers θ_{ab}.

8.2 A sufficient condition for Mn2SB

The frontier census property Mn2SB plays a key role in the classification of connected Γ-colored d-complete posets in the next section. Proposition 8.2.2 gives a sufficient condition for Mn2SB when P is such a poset. This condition is that for all $a \in \Gamma$, the set P_a is unbounded above. Lemma 8.3.1 in the next section shows that the scenario where P satisfies this condition is one of the two primary cases to consider in the classification of connected Γ-colored d-complete posets.

We first give a lemma needed for the proof of this proposition:

Lemma 8.2.1. *Let P be a Γ-colored d-complete poset. Let $\{\mu_a\}_{a\in\Gamma}$ be the component weight function of Proposition 4.4.2. If $(F, I) \in \mathcal{FI}(P)$ is a split in which I contains at least one element of each color, then:*

(a) For all $b \in \Gamma$, we have $\mu_b(F, I) \in \{-1, 0, 1\}$.

(b) For all $b \in \Gamma$, we have $\mu_b(F, I) = +1$ if and only if b is the color of some maximal element of I.

Proof. Let $b \in \Gamma$. By assumption $P_b \cap I \neq \emptyset$, so $\mu_b(F, I) = 1 - \upsilon_b(F, I) \leq 1$. We also have $\mu_b(F, I) \geq -1$ by Theorem 5.3.1 since $\{\mu_a\}_{a\in\Gamma}$ can be used to build an upper P-minuscule representation of $\mathfrak{b}'_+$. Since $\mu_b(F, I) \in \mathbb{Z}$, we get (a). Proposition 4.5.1 shows that $\mu_b(F, I) = +1$ if I has a maximal element of color b. Now suppose $\mu_b(F, I) = +1$. Since $\mu_b(F, I) = 1 - \upsilon_b(F, I)$, we get $\upsilon_b(F, I) = 0$. Thus $P_b \cap I$ has a maximal element y since $\upsilon_b(F, I) = 1$ otherwise. If there are infinitely many elements greater than y in I with colors adjacent to b, then $\upsilon_b(F, I) \geq 1$. Since $\upsilon_b(F, I) = 0$, there are at most finitely many such elements. Suppose there is some $z \in I$ with $y \to z$. By NA we have $b \sim c := \kappa(z)$. Since there are at most finitely many elements greater than y in $P_c \cap I$, we have $z \in \Upsilon_b(F, I)$. Since $-\theta_{cb} > 0$, this contradicts $\sum_{x\in\Upsilon_b(F,I)} -\theta_{\kappa(x),b} = \upsilon_b(F, I) = 0$. Thus y is maximal in I and has color b, so (b) follows. □

Now we give the sufficient condition for Mn2SB when P is a connected Γ-colored d-complete poset. Recall that Γ is also connected by Corollary 8.1.3.

Proposition 8.2.2. *Let P be a connected Γ-colored d-complete poset. Suppose for all $b \in \Gamma$ the set P_b is unbounded above. Then P satisfies Mn2SB.*

Proof. Let $\{\mu_a\}_{a\in\Gamma}$ be the component weight function of Proposition 4.4.2. For the sake of contradiction, suppose P fails Mn2SB. Then there is a color $a \in \Gamma$ and an element x which is minimal in P_a such that either there are infinitely many elements less than x with colors adjacent to a, or there are finitely many such elements and $\sum_{y<x} -\theta_{\kappa(y),a} \geq 3$. In either case, let $\mathcal{S}$ be a finite set of elements less than x with colors adjacent to a such that $\sum_{y\in\mathcal{S}} -\theta_{\kappa(y),a} \geq 3$. Let F be the filter generated by $\mathcal{S}$, and let $I := P - F$. Since $x \in F$ and x is the minimal element of color a, by EC we see that all of P_a is in F. Hence $P_a \cap I = \emptyset$, so we have $\mu_a(F, I) = -1 + \psi_a(F, I)$. Note x is minimal in $P_a \cap F$ since it is minimal in P_a. By local finiteness, the intervals between elements of $\mathcal{S}$ and x are all finite. Thus we have $\Psi_a(F, I) \supseteq \mathcal{S}$ and $\psi_a(F, I) = \sum_{y\in\Psi_a(F,I)} -\theta_{\kappa(y),a}$. Hence $\psi_a(F, I) = \sum_{y\in\Psi_a(F,I)} -\theta_{\kappa(y),a} \geq \sum_{y\in\mathcal{S}} -\theta_{\kappa(y),a} \geq 3$, so $\mu_a(F, I) \geq 2$.

Let z be any element of F and let $c := \kappa(z)$. Let $d \in \Gamma$ be adjacent to c. Since P_d is unbounded above, by AC and local finiteness there must be an element of color d greater than z. Since $z \in F$ and F is a filter, it follows that F contains an element of color d. Continuing in this fashion, since Γ is connected it follows that F contains elements of every color.

Now F is generated by the minimal elements of $\mathcal{S}$. Hence F contains a minimal element of each color by local finiteness. Let $\mathcal{M}$ be the set of minimal elements of each color in F. Let u be minimal in $\mathcal{M}$. So u is minimal in F and $(F, I) \to (F - u, I + u)$ in $\mathcal{FI}(P)$. Let $w \in P$ be such that $u < w$ are consecutive occurrences of the color $\kappa(u)$. Then $(\mathcal{M} - \{u\}) \cup \{w\}$ is the set of minimal elements of each color in $F - u$. Choose a minimal element v of this set and note that v is minimal in $F - u$. Then we get $(F, I) \to (F - u, I + u) \to (F - u - v, I + u + v)$. Continue in this fashion. This process never terminates since $P_b \cap F$ is unbounded above for all $b \in \Gamma$. So there is an infinite chain of covering relations moving upward in $\mathcal{FI}(P)$ starting at (F, I).

Consider the edge colors appearing in this chain in $\mathcal{FI}(P)$. Suppose some colors appear only finitely many times on edges along this chain, including perhaps not at all. Then since Γ is connected and has finitely many nodes, we can find adjacent colors $b \sim c$ such that b appears infinitely many times and c appears only finitely many times (possibly zero) along edges of this chain. By AC all elements of colors b and c in P are comparable, and so $\kappa^{-1}(\{b, c\})$ is totally ordered. Reading off the edge colors appearing in our chain in $\mathcal{FI}(P)$, there is a set of infinitely many consecutive occurrences of the color b that is unbounded above in

$\kappa^{-1}(\{b,c\})$. Thus by local finiteness there cannot be an element of color c above this unbounded set. So all of P_c must lie below some element of color b. But since P_c is also unbounded above, this violates local finiteness. Hence each color must appear infinitely many times on edges along our chain in $\mathcal{FI}(P)$. Thus by transferring elements to ideals along as many edges of each color as necessary, we may always progress far enough upward in this chain to obtain a split whose ideal (and the ideal of every split above it in the chain) contains arbitrarily many elements of each color.

Advance far enough up in this chain to obtain a split whose ideal contains at least one element of each color. By Lemma 8.2.1(a) the values taken by $\{\mu_a\}_{a\in\Gamma}$ at this split must all be in $\{-1,0,1\}$. Hence there are $3^{|\Gamma|}$ possibilities for this Γ-set of values. This is true for every split above it in the chain as well. Since this is an infinite chain, it contains some infinite subchain $(F_1,I_1) < (F_2,I_2) < \cdots$ such that $\mu_b(F_k,I_k) = \mu_b(F_j,I_j)$ for all $b \in \Gamma$ and all $k, j \geq 1$. Let m be the number of edges in the original chain between (F,I) and (F_1,I_1). Choose N large enough so that I_N contains at least $m+1$ elements of each color.

Consider the initial portion $(F,I) \to \cdots \to (F_1,I_1)$ of length m of our fixed chain in $\mathcal{FI}(P)$. Take the path in $\mathcal{FI}(P)$ from (F_1,I_1) to (F,I) moving downward along these edges. Let $x_1,\ldots,x_m$ with respective colors $a_1,\ldots,a_m$ be the elements transferred from ideals to filters along the edges of this path. For every $1 \leq j \leq m$, the element x_j is maximal in $I_1-x_1-\cdots-x_{j-1}$. Thus we have $\mu_{a_j}(F_1+x_1+\cdots+x_{j-1}, I_1-x_1-\cdots-x_{j-1}) = +1$ by Proposition 4.5.1(iii).

We now produce a path in $\mathcal{FI}(P)$ descending from (F_N,I_N). Start out by noting that $\mu_{a_1}(F_N,I_N) = \mu_{a_1}(F_1,I_1) = +1$. We can apply Lemma 8.2.1(b) since I_N has an element of every color. Hence I_N has a maximal element y_1 of color a_1. Note that $\Delta_c[(F_N+y_1, I_N-y_1),(F_N,I_N)] = -|P_c \cap \{y_1\}| = -|P_c \cap \{x_1\}| = \Delta_c[(F_1+x_1, I_1-x_1),(F_1,I_1)]$ for all $c \in \Gamma$. Applying Equation (3.6) to the component weight function $\{\mu_a\}_{a\in\Gamma}$ and using this observation, for all $b \in \Gamma$ we get

$$\begin{aligned}\mu_b(F_N+y_1, I_N-y_1) - \mu_b(F_N,I_N) &= \sum_{c\in\Gamma} \theta_{cb}\Delta_c[(F_N+y_1,I_N-y_1),(F_N,I_N)]\\ &= \sum_{c\in\Gamma} \theta_{cb}\Delta_c[(F_1+x_1,I_1-x_1),(F_1,I_1)] = \mu_b(F_1+x_1,I_1-x_1) - \mu_b(F_1,I_1).\end{aligned}$$

Since $\mu_b(F_N,I_N) = \mu_b(F_1,I_1)$, we get $\mu_b(F_N+y_1,I_N-y_1) = \mu_b(F_1+x_1,I_1-x_1)$ for all $b \in \Gamma$. Applying this conclusion to the color a_2 gives $\mu_{a_2}(F_N+y_1,I_N-y_1) = \mu_{a_2}(F_1+x_1,I_1-x_1) = +1$. We can again apply Lemma 8.2.1(b) since $I_N - y_1$ has an element of every color. Thus we see that $I_N - y_1$ has a maximal element

y_2 of color a_2. Iterating this reasoning produces our desired path of length m moving downward from (F_N, I_N) by transferring elements $y_1, \ldots, y_m$ from ideals to filters along edges with respective colors $a_1, \ldots, a_m$. (Note that N was chosen large enough so that Lemma 8.2.1(b) can be applied to any split that can be reached from (F_N, I_N) by a path of length m or less.) Set $(F', I') := (F_N + y_1 + \cdots + y_m, I_N - y_1 - \cdots - y_m)$. Recall that $(F, I) = (F_1 + x_1 + \cdots + x_m, I_1 - x_1 - \cdots - x_m)$. Thus we have $\mu_b(F', I') = \mu_b(F, I)$ for all $b \in \Gamma$. Since I' contains at least one element of each color by our choice of N, we see that $\mu_b(F', I') \in \{-1, 0, 1\}$ for all $b \in \Gamma$ by Lemma 8.2.1(a). Hence $\mu_b(F, I) \in \{-1, 0, 1\}$ for all $b \in \Gamma$. But this contradicts $\mu_a(F, I) \geq 2$ for our fixed color a. $\square$

8.3 Classification of connected Γ-colored d-complete posets

In Theorem 8.3.8 we classify all connected Γ-colored d-complete posets. By Corollary 8.1.4 this will classify all Γ-colored d-complete posets. Recall that if P is a connected poset satisfying EC, NA, and AC, then Corollary 8.1.3 implies that Γ is connected.

The next two properties for a Γ-colored poset P pertain to whether the elements of all color chains are bounded above or are unbounded above:

(CBA): For all $a \in \Gamma$, the color chain P_a is bounded above.

(CUA): For all $a \in \Gamma$, the color chain P_a is unbounded above.

One of these must hold when P satisfies most of the properties required to be Γ-colored d-complete:

Lemma 8.3.1. *Let P be a connected poset that satisfies EC, NA, AC, and MxFGA. Then P satisfies either CBA or CUA.*

Proof. Suppose P does not satisfy CBA or CUA. Since Γ is connected, we can find adjacent colors $a \sim b$ such that P_a is bounded above and P_b is unbounded above. Let x be the maximal element of color a. By AC all elements of P_b are comparable to x. By local finiteness, since P_b is unbounded above there must be at least one element of color b above x. Thus there are infinitely many elements of color b above x. But this violates MxFGA. $\square$

Since Mx1SB implies MxFGA, this lemma applies to Γ-colored d-complete posets. We use Proposition 7.5.3 to handle the CBA case:

Proposition 8.3.2. *Suppose P is a connected Γ-colored d-complete poset. If P satisfies CBA, then P is finite.*

Since connected finite Γ-colored d-complete posets are dominant minuscule heaps, for the CBA case we can now apply the classifications found by Proctor and Stembridge. The idea behind the proof was suggested by Proctor.

Proof. Since P satisfies CBA, we may define T to be the set of maximal elements of each color. For the sake of contradiction suppose P is infinite. Every finite filter of P containing T is also Γ-colored d-complete. By Corollary 7.4.6, every finite filter of P containing T is a dominant minuscule heap. Define $F_1 := T$ and note by Mx1SB that F_1 is a (finite) filter of P. Let $F_1 \subset F_2 \subset \cdots$ be an infinite sequence of distinct finite filters of P. Each of these filters is a connected dominant minuscule heap with top tree T. Now fix some $k \geq 1$ and consider F_k. We know F_k must be a slant sum of slant irreducible dominant minuscule heaps. By Proposition 7.5.3, there are at most $|T|$ slant irreducible dominant minuscule heaps in the slant sum making up F_k. Let N be the cardinality of the largest possible slant irreducible dominant minuscule heap given a top tree of cardinality $|T|$. This number can be determined by examining the seventeen families of slant irreducible dominant minuscule heaps referenced in Proposition 7.5.3; we only use its finiteness. Then the cardinality of F_k is bounded by $N \cdot |T|$. Since $k \geq 1$ was arbitrary, this shows that the cardinalities of the filters in this sequences are bounded by $N \cdot |T|$. This contradicts that the filters $F_1, F_2, \ldots$ are distinct, so P is finite. □

Otherwise a connected Γ-colored d-complete poset P satisfies CUA. By Proposition 8.2.2, here we know that P satisfies Mn2SB. This implies P satisfies MnFLA. The dual of Lemma 8.3.1 now shows that P satisfies one of the following two properties:

(CBB): For all $a \in \Gamma$ the color chain P_a is bounded below.

(CUB): For all $a \in \Gamma$ the color chain P_a is unbounded below.

So we now split the CUA case into two subcases, those of CBB and CUB. We use Corollary 7.4.4 when P satisfies CUA and CUB:

Proposition 8.3.3. *Let P be a connected Γ-colored d-complete poset. Suppose P satisfies CUA and CUB. Then P is a full heap.*

Proof. We know that for all $a \in \Gamma$ the set P_a is isomorphic to $\mathbb{Z}$, so P satisfies G3. Then Corollary 7.4.4 shows P is a full heap. □

The remaining case occurs when P satisfies CUA and CBB. This case is more complicated; it is resolved in Corollary 8.3.7.

Lemma 8.3.4. *Let P be a connected Γ-colored d-complete poset. If P satisfies CUA and CBB, then P does not satisfy Mn1SB.*

Proof. Suppose P satisfies Mn1SB. Then the order dual poset P^* satisfies Mx1SB and hence is Γ-colored d-complete as well. Since P satisfies CBB, we know P^* satisfies CBA. Then P^* is an infinite Γ-colored d-complete poset satisfying CBA, contradicting Proposition 8.3.2. □

The notation established in the following discussion will be maintained through the proof of Lemma 8.3.6. Continue to assume P is a connected Γ-colored d-complete poset that satisfies CUA and CBB. We now describe an iterative process which will produce a new poset Q. The poset Q will be shown to be a full heap in Lemma 8.3.6, and P will be a filter of Q.

By Proposition 8.2.2 we know P satisfies Mn2SB. By Lemma 8.3.4 we know P does not satisfy Mn1SB. Thus there is some color $a \in \Gamma$ and minimal element $y \in P_a$ such that $\sum_{z<y} -\theta_{\kappa(z),a} = 2$. There are two possibilities:

(i) There are exactly two elements u and v less than y with colors adjacent to a. The colors of u and v are both 1-adjacent to a.

(ii) There is exactly one element w less than y with color adjacent to a. The color of w is 2-adjacent to a.

In either case, create the symbol x and define x to have color a. Set $Q_1 := P \cup \{x\}$. Define a partial order on Q_1 by inheriting the partial order on P and the following covering relation(s) depending upon whether Case (i) or Case (ii) listed above applies:

(i) If $u < v$ (respectively $v < u$), then add the covering relation $x \to u$ (respectively $x \to v$). If u and v are incomparable, then add the covering relations $x \to u$ and $x \to v$.

(ii) Add the covering relation $x \to w$.

This "downward extension" process is similar to that used in [Pr3] for the classification of finite uncolored d-complete posets.

Lemma 8.3.5. *The poset Q_1 is a connected Γ-colored d-complete poset. It satisfies CUA and CBB.*

Proof. It is obvious that Q_1 is connected and satisfies CUA and CBB. Thus it satisfies Mx1SB vacuously. It is easy to see that Q_1 has the properties EC, NA, and AC. Now consider the property I2$\vee$1A. The only interval between consecutive elements of the same color in Q_1 that is not contained in P is the interval $[x, y]$. By construction, this interval contains u and v in Case (i) and it contains w in Case (ii). It does not contain any other elements with colors adjacent to a. In Case (i) we see this interval satisfies I2$\vee$1A(i), and in Case (ii) we see this interval satisfies I2$\vee$1A(ii). Hence I2$\vee$1A is satisfied. $\square$

So Proposition 8.2.2 and Lemma 8.3.4 both apply to Q_1 as well. Thus the construction that led to Q_1 can be repeated indefinitely to construct an infinite sequence of posets $Q_2, Q_3, \ldots$ all of which are connected Γ-colored d-complete posets satisfying CUA and CBB. We have $P \subset Q_1 \subset Q_2 \subset \cdots$, and each poset is a filter of the next. Let Q be the poset resulting from infinitely iterating this construction. So P is a filter of Q.

Lemma 8.3.6. *The poset Q is a connected full heap.*

Proof. We first show Q is locally finite. Let $x, y \in Q$ with $x < y$. Since $x \in Q_k$ for some $k \geq 1$ and Q_k is a filter of Q, we have $[x, y] \subseteq Q_k$. Since P is locally finite and $Q_k - P$ is finite, we see Q_k is also locally finite. Thus $[x, y]$ is finite, and so Q is locally finite.

Now we show Q is connected. Let $x, y \in Q$. Then $x, y \in Q_k$ for some $k \geq 1$. Since Q_k is connected, there is a finite path from x to y in Q_k. This is a path from x to y in Q as well, so Q is connected.

The poset Q satisfies EC, NA, AC, and I2$\vee$1A: Any occurrence of two neighbors or elements with equal or adjacent colors in Q occurs in one of the filters $P, Q_1, Q_2, \ldots$, and each of these filters satisfies these properties. Suppose there is some color in which there were only finitely many elements of that color added to form Q. Then since Γ is connected, we can choose adjacent colors $a \sim b$ such that finitely many elements of color a and infinitely many elements of color b were added to form Q. By AC the last element x of color a added to form Q is comparable to every element of color b in Q. Since Q is locally finite, there must be infinitely many elements of color b below x. This implies there is some $j \geq 1$ such that Q_j fails Mn2SB. This violates Proposition 8.2.2 applied to Q_j. Thus there are infinitely many elements of each color added to form Q. Hence all color chains Q_a for $a \in \Gamma$ are isomorphic to $\mathbb{Z}$ as posets, so Q satisfies G3. Corollary 7.4.4 then shows Q is a full heap. $\square$

Thus we have finished the case where P satisfies CUA and CBB:

Corollary 8.3.7. *Let P be a connected Γ-colored d-complete poset. If P satisfies CUA and CBB, then P is the filter of some connected full heap.*

We now present the classification of connected Γ-colored d-complete posets:

Theorem 8.3.8. *The connected* Γ*-colored d-complete posets are classified as follows:*

(a) The connected finite Γ*-colored d-complete posets are exactly the connected dominant minuscule heaps of [Ste]. Hence the connected finite* Γ*-colored d-complete posets are slant sums consisting of posets taken from the fifteen slant irreducible families in [Pr3] and the two slant irreducible families in [Ste]. These slant sums are described in Proposition 7.5.3 and the possible Dynkin diagrams for such slant sums are described in Corollary 7.5.4.*

(b) The connected infinite Γ*-colored d-complete posets are exactly the filters of the connected full heaps of [Gr3], which were classified [Gr3, McG] with the list given in Theorem 6.2.2 of [Gr3].*

Proof. Let P be a connected Γ-colored d-complete poset. First suppose P is finite. Corollary 7.4.6, Proposition 7.5.3, and Corollary 7.5.4 finish Part (a).

Now suppose P is an infinite connected Γ-colored d-complete poset. Lemma 8.3.1 shows P satisfies CBA or CUA. Proposition 8.3.2 shows P must satisfy CUA. Proposition 8.2.2 shows P must satisfy Mn2SB, and hence MnFLA. The dual of Lemma 8.3.1 shows P satisfies CUB or CBB. If P satisfies CUB, then Proposition 8.3.3 shows P is a connected full heap. If P satisfies CBB, then Corollary 8.3.7 shows P is the filter of some connected full heap.

Conversely, filters of full heaps still satisfy properties G1, G2, and G5. Thus filters of connected full heaps satisfy EC, NA, AC, and I2$\vee$1A by Corollary 7.4.3. They satisfy Mx1SB vacuously, so they are connected infinite Γ-colored d-complete posets. $\square$

The classification gives the following uniqueness result for the $\mathfrak{h}'$-weight of an upper P-minuscule representation of $\mathfrak{b}'_+$, after showing there is only one nontrivial component of $\mathcal{FI}(P)$:

Corollary 8.3.9. *Suppose P is a connected* Γ*-colored d-complete poset.*

(a) Then $\mathcal{FI}(P)$ *has exactly one component* $\mathcal{C}$ *that contains more than one split.*

(b) The μ*-diagonal operators are the unique operators satisfying Part (i) of Theorem 5.3.1 on* $\mathcal{C}$*.*

Green defined [Gr3, Def. 3.2.1] a *proper ideal* of a poset P to be an ideal I for which $P_a \cap I \neq \emptyset$ and $P_a \cap I \neq P_a$ for all $a \in \Gamma$. Section 3.2 of [Gr3] develops several results for proper ideals, two of which we use here.

Proof. First suppose P is finite. Then $\mathcal{FI}(P)$ has one component by Proposition 3.2.2. Otherwise P is infinite. By Theorem 8.3.8(b), we see P must be a filter of a connected full heap Q. Lemma 3.2.4(v) of [Gr3] states that every ideal of Q is proper except for $\emptyset$ and Q. In our language, Proposition 3.2.12(ii) of [Gr3] then implies that there is a finite path in $\mathcal{FI}(Q)$ between any two splits whose ideals are proper ideals of Q. Using these results, it follows that $\mathcal{FI}(Q)$ consists of three components: The component $\{(Q, \emptyset)\}$, the component $\{(\emptyset, Q)\}$, and the component containing every other split. Now note that $\mathcal{FI}(P)$ can be realized inside of $\mathcal{FI}(Q)$ with the the principal filter generated by the split $(P, Q - P)$. Hence $\mathcal{FI}(P)$ contains at most three components, with only one component that contains more than one split. Thus we get (a). Let C be the nontrivial component of $\mathcal{FI}(P)$. Since κ is surjective, there is an edge of every color in $\mathcal{FI}(P)$. The trivial components do not have any edges, so we see C contains an edge of every color. Then (b) follows by combining the last statement of Theorem 5.3.1 and Corollary 4.5.2. $\square$

We now show that a Γ-colored d-complete poset satisfies AN; see Proposition 3.1.5 for the definition of this property. This implies a Γ-colored minuscule poset also satisfies AN. By Proposition 3.1.5 in the simply laced case and Proposition 3.3.5 in the general case, this will allow us to conclude that the bracket of two non-commuting color raising operators must be nonzero for upper P-minuscule representations and P-minuscule representations built from Γ-colored posets. We handle the finite case first:

Corollary 8.3.10. *Let P be a finite Γ-colored d-complete poset. Then P satisfies AN.*

We use only the properties NA, AC, and S4 in the proof.

Proof. Suppose P fails AN. Then there are adjacent colors $a, b \in \Gamma$ such that there are no neighbors in P with colors a and b. Since κ is surjective, choose elements $x \in P_a$ and $y \in P_b$. By AC these elements are comparable in P. Without loss of generality, assume $x < y$ and that there are no elements of colors a or b in (x, y). By assumption we know y does not cover x. Choose a saturated chain $x \to u \to \cdots \to y$. By NA this chain induces a path from a to b in Γ that contains at least three colors. The colors a and b appear in this path only at the endpoints. This forms a cycle in Γ since $a \sim b$. But P is a dominant minuscule heap by Corollary 7.4.6, so this contradicts S4. $\square$

Now we handle the general cardinality case as well by making use of Green's property G4:

Corollary 8.3.11. *Let P be a Γ-colored d-complete poset. Then P satisfies AN.*

Proof. Let $a, b \in \Gamma$ be adjacent. These colors are in the same connected component Γ' of Γ. Let P' be the connected component of P that is colored by Γ' guaranteed by Proposition 8.1.2. By Corollary 8.1.4(a) we know P' is a Γ'-colored d-complete poset. If P' is finite, then by Corollary 8.3.10 we are done. Otherwise, note that P' is the filter of a full heap Q by Theorem 8.3.8(b). Then P'_a is unbounded above, so let $u, v \in P'_a$ be consecutive occurrences of the color a with $u < v$. Suppose there is an element $z \in Q$ such that $z \to v$. By Corollary 7.4.3 we know that Q satisfies NA and AC, and so $\kappa(z) \sim a$. Thus by AC we have $z \in (u, v)$. Since $u \in P'$ and P' is a filter of Q with $z > u$, we know that $z \in P'$ also. Clearly if $w \in Q$ with $v \to w$, then $w \in P'$ as well. Thus all neighbors of v in Q are also in P'. Since Q is a full heap, it satisfies G4. Thus at least one neighbor of v must have color b. Hence P' satisfies AN, and hence so does P. □

8.4 Classification of connected Γ-colored minuscule posets

In Theorem 8.4.5 we classify the connected Γ-colored minuscule posets. By Corollary 8.1.4 this will classify all Γ-colored minuscule posets.

We first handle the case when P is finite. We need the statement about the unique minimal element made in the following lemma:

Lemma 8.4.1. *Suppose P is a finite connected poset satisfying NA, AC, I2∨1A, and Mx1SB. Then P has a unique maximal element. Hence a finite connected Γ-colored d-complete poset has a unique maximal element and a finite connected Γ-colored minuscule poset has a unique maximal element and a unique minimal element.*

Proof. Proposition F2 of [Pr3] (restated as Theorem 4.4(a) in [PrSc]) said that a finite connected poset has a unique maximal element if whenever an element is covered by two other elements, there is a fourth element covering those two. Let $s, u, v \in P$ be such that $s \to u$ and $s \to v$ with $u \neq v$. Define $c := \kappa(s)$. By NA we know that $\kappa(u) \sim c$ and $\kappa(v) \sim c$. So by Mx1SB we know s is not the maximal element of color c. Let $t \in P$ be such that $s < t$ are consecutive occurrences of the color c. By AC we know that $u, v \in (s, t)$. If t does not cover one of u or v, then there is another element x not equal to u or v such that $x \to t$. By NA we would have $\kappa(x) \sim c$, violating I2∨1A. Thus t covers u and v. Applying Proposition F2 of [Pr3], we see that P has a unique maximal element. A finite connected Γ-colored d-complete poset satisfies NA, AC, I2∨1A, and Mx1SB. A finite connected Γ-colored minuscule poset and its order dual both satisfy these properties. □

Every finite Γ-colored minuscule poset is Γ-colored d-complete. So by Corollary 7.4.6 we know such a poset is a dominant minuscule heap. In the following proof we use Stembridge's axiom S3.

Lemma 8.4.2. *Suppose P is a finite connected Γ-colored minuscule poset. Then either*

(i) the poset P is a chain and Γ is simply laced, or

(ii) the poset P is slant irreducible.

The singleton poset is the only poset that occurs in both (i) and (ii). The only chains with two or more elements that are slant irreducible have Dynkin diagrams of finite type C, so they are the chains that occur in (ii).

Proof. Suppose P is not slant irreducible as a dominant minuscule heap. Let T be the top tree in P; i.e. the set of maximal elements of each color. Then there are $w, z \in T$ with $w \to z$ such that z is the only element of its color in P. Let $x, y \in T$ with $x \to y$ be such that x is a minimal element in such a situation, and set $b := \kappa(y)$. By NA we have $b \sim a := \kappa(x)$. By AC every occurrence of the color a must be comparable to y. Then Mn1SB shows that x must be the minimal occurrence of the color a in P. But since $x \in T$, it is also the maximal occurrence of the color a. Thus x is the only occurrence of the color a in P. Also, by possibly repeated applications of S3, we see that the filter generated by x is a chain contained in T.

Suppose there exists some $u \in P$ such that $u \to x$ and set $c := \kappa(u)$. The minimality of the choice of x above shows $u \notin T$. Thus let $v \in P_c$ be such that $u < v$ are consecutive occurrences of the color c. By NA we have $c \sim a$, so we have $x \in [u, v]$ by AC. By [Ste, Prop 3.3] we see that $[u, v]$ is either a double-tailed diamond or a chain. This proposition implies $[u, v]$ has an odd number of ranks and is colored symmetrically by rank. Since x is the only element of color a in P, it must be in the middle rank of $[u, v]$. Since the filter generated by x is a chain, only y covers x in P. Thus y must be in $[u, v]$ as well, and y cannot be in the middle rank of $[u, v]$. But since $[u, v]$ is symmetrically colored by rank, this contradicts that y is the only element of color b in P. Thus x is minimal in P. By Lemma 8.4.1 it is the unique minimal element of P. So the filter generated by x is all of P. Recall that the filter generated by x is a chain contained in T. This shows P is a chain and $P = T$.

For the sake of contradiction suppose Γ is multiply laced. Then there are colors e and f such that e is k-adjacent to f for some $k \geq 2$. Since $P = T$, let s and t be the unique elements in P with colors e and f, respectively. If $s < t$, then P fails Mn1SB since t is the minimal element of color f in P. Likewise, if $t < s$, then P fails Mx1SB since t is the maximal element of color f in P. Thus Γ is simply laced. □

Using the fact that both a Γ-colored minuscule poset and its order dual are Γ-colored d-complete, we can now finish the finite case. We recalled the definition of the colored minuscule posets of Proctor in Section 7.1.

These posets were listed in [Pr1] and appear in Appendix B of [McG]. They are the "principal subheaps" of Green [Gr3, Def. 5.5.3].

Proposition 8.4.3. *Let P be a finite connected poset.*

(a) Suppose P is a Γ-colored minuscule poset. Then P is a colored minuscule poset.

(b) Suppose P is a colored minuscule poset. Let Γ be the Dynkin diagram of the Lie type associated to P. Then P is a Γ-colored minuscule poset.

Proof. To prove (a), set $n := |\Gamma|$. By Lemma 8.4.2 we know P is a chain and Γ is simply laced, or P is slant irreducible. First suppose that P is a chain and Γ is simply laced. Any interval between two consecutive elements of the same color in Γ must be a double-tailed diamond by [Ste, Prop. 3.3]. Hence the colors of all elements in P must be distinct, so $|P| = n$. So as an uncolored poset, the poset P is isomorphic to the uncolored minuscule posets $a_n(1)$ and $a_n(n)$. By Corollary 7.4.6 and Fact 7.5.1(a) we know P is a colored d-complete poset. Since P is a colored d-complete poset that as an uncolored poset is isomorphic to an uncolored minuscule poset, by Fact 7.1.1 it is a colored minuscule poset.

Otherwise P is slant irreducible. Note that Lemma 8.4.2 can be applied to the order dual P^* since it is Γ-colored minuscule; thus P^* is also a finite slant irreducible Γ-colored d-complete poset. First suppose that Γ is simply laced. Then by Proposition 7.5.3 we know that both P and P^* belong to one of the fifteen families of slant irreducible posets in [Pr3]. We survey these fifteen families to find posets P such that both P and P^* are slant irreducible d-complete posets. The discussion following Fact 7.5.1 explains how Proctor's notion of top tree produces the same elements of P as our notion of top tree. His fifteen families are arranged according to their uncolored top trees. Here the top tree of P has cardinality n. The posets we list here are the uncolored minuscule posets (as in Section 7.1). Family 1 contains $a_n(j)$ for $1 < j < n$, and no others. Family 2 contains $d_n(n)$ and $d_n(n-1)$ for $n \geq 4$, as well as $d_4(1)$, and no others. Family 4 contains $d_n(1)$ for $n \geq 5$, and no others. Family 8 contains $e_6(1)$ and $e_6(6)$, and no others. Family 15 is $e_7(7)$ and contains no others. Families 3, 5–7, and 9–14 contain none. Repeating the reasoning above that was applied to $a_n(1)$ and $a_n(n)$, we see that these posets are colored minuscule posets.

Now suppose Γ is multiply laced. We use the classification of slant irreducible dominant minuscule heaps from [Ste]. We survey his two additional families of slant irreducible dominant minuscule heaps as above, noting that these posets are already colored by Dynkin diagrams. The first additional slant irreducible family contains $b_n(n)$ for $n \geq 2$, and no others. Note that $c_2(1) = b_2(2)$, so now restrict to $n \geq 3$. Then the second additional slant irreducible family contains $c_n(1)$ for $n \geq 3$, and no others. It can be seen that these colorings match the colorings specified by Theorem 11 of [Pr1] and shown in Appendix B of [McG].

Part (b) can be readily confirmed on a case-by-case basis by checking the colored posets displayed in Appendix B of [McG] for the defining properties of a Γ-colored minuscule poset. □

We can now finish the infinite case. We recalled the definition of the full heaps of Green in Section 7.3. These posets appear in Appendix C of [McG].

Proposition 8.4.4. *Let P be an infinite connected poset colored by a Dynkin diagram Γ. The following are equivalent:*

(i) The poset P is a Γ-colored minuscule poset.

(ii) The poset P is a full heap.

Proof. Assume that (i) holds. We know both P and the order dual P^* are Γ-colored d-complete. So by Theorem 8.3.8(b) they are both filters of full heaps. Thus for every $a \in \Gamma$ the set P_a is isomorphic as a poset to $\mathbb{Z}$, so P satisfies G3. Corollary 7.4.4 then shows P is a full heap.

Now assume (ii) holds. Then P satisfies Mx1SB and Mn1SB vacuously. Corollary 7.4.3 shows that P also satisfies EC, NA, AC, and I2$\vee$1A. □

Combining Propositions 8.4.3 and 8.4.4 gives the classification of connected Γ-colored minuscule posets:

Theorem 8.4.5. *The connected Γ-colored minuscule posets are classified as follows:*

(a) The finite connected Γ-colored minuscule posets are exactly the minuscule posets of [Pr1] after they have been colored by [Pr1, Thm. 11]. Thus [Pr1] gives the complete list of connected finite Γ-colored minuscule posets.

(b) The infinite connected Γ-colored minuscule posets are exactly the connected full heaps of [Gr3], which were classified by Green [Gr3, Thm. 6.6.2] and McGregor-Dorsey [McG, Thm. 4.7.1]. Hence Theorem 6.6.2 of [Gr3] gives the complete list of connected infinite Γ-colored minuscule posets.

8.5 Classification of upper P-minuscule and P-minuscule representations

We now apply Theorem 6.1.1, Corollary 8.1.4, and the classifications of Theorems 8.3.8 and 8.4.5 to obtain the classifications of all upper P-minuscule representations of $\mathfrak{b}'_+$ and P-minuscule representations of $\mathfrak{g}'$. This result is the culmination of this dissertation; it solves our central problem of filling in Table 1.1.

Theorem 8.5.1. *The upper P-minuscule and P-minuscule representations are classified as follows:*

(a) The upper P-minuscule representations of $\mathfrak{b}'_+$ *are precisely indexed by direct sums of finitely many posets taken from the lists in Theorem 8.3.8 of connected* Γ*-colored d-complete posets.*

(b) The P-minuscule representations of $\mathfrak{g}'$ *are precisely indexed by direct sums of finitely many posets taken from the lists in Theorem 8.4.5 of connected* Γ*-colored minuscule posets.*

In both parts, the Dynkin diagrams Γ *are the disjoint unions of the connected Dynkin diagrams that color the posets making up these direct sums.*

CHAPTER 9

Further remarks

We first discuss how our foremost characterization result for P-minuscule representations of $\mathfrak{g}'$, Theorem 5.4.2, compares to Green's Theorem 4.1.6(i) of [Gr3]. Then we use another result of Green to indicate how most of our P-minuscule representations can be extended to the full Kac–Moody algebra $\mathfrak{g}$. Finally, we include a proposed new definition of abstract minuscule representation for Kac–Moody algebras.

9.1 Comparison of Theorem 6.1.1(b) to Green's Theorem 4.1.6(i)

Let $\mathfrak{g}'$ be a derived Kac–Moody algebra with Dynkin diagram Γ. Let P be a connected full heap over Γ. Let φ_1 be the representation of $\mathfrak{g}'$ constructed in Theorem 4.1.6(i) of [Gr3]. By Theorem 8.4.5(b) we see that P also arises as a Γ-colored minuscule poset. The "if" direction of Theorem 6.1.1(b) then shows P can be used to build a P-minuscule representation φ_2 of $\mathfrak{g}'$. Here Green's representation φ_1 of $\mathfrak{g}'$ can be obtained from our P-minuscule representation φ_2 of $\mathfrak{g}'$ by removing the splits $(P, \emptyset)$ and $(\emptyset, P)$ from our $\mathcal{FI}(P)$.

9.2 Extension of P-minuscule representations to the full Kac–Moody algebra

When P is a finite connected Γ-colored minuscule poset, Theorem 8.4.5 shows that Γ has finite Lie type and so $\mathfrak{g}' = \mathfrak{g}$. Can the representations built from full heaps be extended from $\mathfrak{g}'$ to the full Kac–Moody algebra $\mathfrak{g}$ for Γ? For full heaps of untwisted affine type, Theorem 7.10 of [Gr1] did this. Here we use this result to extend Theorem 6.1.1(b) from $\mathfrak{g}'$ to $\mathfrak{g}$ for most of our P-minuscule representations. We say $\mathcal{FI}(P)$ *carries a representation of* $\mathfrak{g}$ if a representation of $\mathfrak{g}'$ carried by $\mathcal{FI}(P)$ can be extended to a representation of $\mathfrak{g}$. Such a representation is *P-minuscule* if the restriction to $\mathfrak{g}'$ is P-minuscule.

Theorem 9.2.1. *Let P be a poset whose elements are colored by the nodes of a connected Dynkin diagram* Γ*. Suppose* Γ *does not have twisted affine type. Let* $\mathcal{FI}(P)$ *be the lattice of filter-ideal splits of P. The following are equivalent:*

(i) The lattice $\mathcal{FI}(P)$ *carries a P-minuscule representation of* $\mathfrak{g}$*.*

(ii) The poset P is a Γ*-colored minuscule poset.*

Proof. Restrict from $\mathfrak{g}$ to $\mathfrak{g}'$ and apply Theorem 6.1.1(b) to see that (i) implies (ii). Now let P be Γ-colored minuscule. Apply Theorem 6.1.1(b) to obtain a P-minuscule representation of $\mathfrak{g}'$. Then use Theorem 8.4.5: If P is finite, then it must be one of the colored minuscule posets of [Pr1] and so $\mathfrak{g} = \mathfrak{g}'$. If P is infinite, then it is a full heap. Since Γ does not have twisted affine type, it must have untwisted affine type by Theorem 4.7.1 of [McG]. Extend from $\mathfrak{g}'$ to $\mathfrak{g}$ using Theorem 7.10 of [Gr1]. $\square$

This result can be applied whenever Γ is simply laced since the twisted affine Dynkin diagrams are each multiply laced. We have worked with Green (personal communication) to confirm that the above representations of $\mathfrak{g}$ built from full heaps are irreducible. These are the infinite dimensional P-minuscule representations of $\mathfrak{g}$ after the two trivial components respectively spanned by the vectors $\langle P, \emptyset\rangle$ and $\langle \emptyset, P\rangle$ for the splits $(P, \emptyset)$ and $(\emptyset, P)$ are discarded.

9.3 Abstract minuscule representations

A *weight representation* of a Kac–Moody algebra $\mathfrak{g}$ is one for which the actions of the elements of its Cartan subalgebra $\mathfrak{h}$ can be simultaneously diagonalized. It is *multiplicity free* if each of the resulting $\mathfrak{h}$-weight spaces is 1-dimensional. The weights inherit the usual simple root partial order from $\mathfrak{h}^*$. An irreducible highest weight representation of a semisimple Lie algebra $\mathfrak{g}$ is *minuscule* if each of its weights is in the Weyl group orbit of the highest weight. No such highest weight representations exist when $\mathfrak{g}$ is infinite dimensional. Is it nonetheless possible to develop a meaningful notion of "minuscule" representation for the infinite dimensional Kac–Moody algebras? Green's doubly infinite representations of certain affine Lie algebras built from full heaps are the prime candidates to be regarded as the minuscule representations of the infinite dimensional Kac–Moody algebras. These representations and the minuscule representations of the semisimple Lie algebras are multiplicity free weight representations, their $\mathfrak{h}$-weights form a connected poset under the order on $\mathfrak{h}^*$, and their $\mathfrak{h}'$-weights have values only in the set $\{-1, 0, 1\}$ for the simple coroot actions. Proctor has proposed that these three axioms should be taken as the definition of a "minuscule" representation of any Kac–Moody algebra. It is hoped that it can be shown that when such a representation is restricted to $\mathfrak{g}'$, it is essentially a P-minuscule representation carried by the lattice $\mathcal{FI}(P)$ of splits for a Γ-colored poset P. One could then apply the "necessary" direction of Theorem 6.1.1(b) to see that P is a Γ-colored minuscule poset. Then Theorem 8.5.1(b) would be used to list all such representations.

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